

LECTURE NOTES  
ON  
NONEXPANSIVE AND MONOTONE MAPPINGS  
IN  
BANACH SPACES

by

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## P R E F A C E

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## T a b l e o f C o n t e n t s

I. <u>Preliminaries</u>	
1. Uniformly convex Banach spaces.....	1
2. Strictly convex Banach spaces.....	5
3. Convex sets with normal structure.....	6
4. Dual spaces and weak topology.....	8
5. Reflexive Banach spaces.....	15
6. Hilbert space structure in finite dimensional linear spaces.....	17
7. Adjoint mappings.....	18
References.....	20
II. <u>Nonexpansive mappings in Banach spaces</u>	
1. Nonexpansive mappings.....	22
2. Isometric mappings.....	28
3. Common fixed points of commuting nonexpansive mappings.....	31
4. Nonexpansive mappings and successive approximations....	34
5. Demicompact nonexpansive mappings and successive approximations.....	38
6. Weak convergence of successive approximations.....	47
7. Contractive approximations of nonexpansive mappings....	51
8. Extensions of nonexpansive mappings.....	53
References.....	59
III. <u>Monotone mappings in Banach spaces</u>	
1. Introduction.....	62
2. Monotone sets.....	68
3. Monotone mappings.....	73
4. Examples of monotone mappings.....	83
5. Coercive mappings.....	88
6. Inequalities for monotone mappings.....	92
7. Surjectivity property of monotone mappings.....	101
8. Nonexpansive and monotone mappings in Hilbert spaces...	105

9. Semimonotone mappings.....	107
10. Densely defined monotone mappings.....	109
References.....	114

#### IV. Complex monotone mappings

1. Complex monotone and complex coercive mappings.....	120
2. Surjectivity property of complex monotone mappings....	123
References.....	128

# C h a p t e r I

## P R E L I M I N A R I E S

This introductory chapter is devoted to a short recall of some advanced concepts and theorems from functional analysis, centered around the compactness properties of bounded convex sets in reflexive Banach spaces and geometric properties of uniformly convex and strictly convex Banach spaces. Fundamental and well-known results are stated without proofs which can easily be found in standard textbooks of functional analysis, e.g. in M. M. Day [4], G. Koethe [6] or K. Yosida [8]. Little-known technical lemmas and some novel results which can be found only in original papers are provided with proofs.

### 1. Uniformly convex Banach spaces.

In a given real or complex Banach space  $X$ ,  $B(x,r)$  and  $S(x,r)$  will denote, respectively, the ball and the sphere centered at  $x$  of radius  $r$ ,

$$B(x,r) = \{y \in X : \|x-y\| \leq r\}; \quad S(x,r) = \{y \in X : \|x-y\| = r\}.$$

Definition 1.1 (Clarkson [3]). A Banach space  $X$  is called uniformly convex (uniformly rotund, in the terminology of [4]) if for any  $\varepsilon > 0$  there is a  $\delta = \delta(\varepsilon) > 0$  such that if  $\|x\| = \|y\| = 1$  and  $\|x-y\| \geq \varepsilon$ , then  $\|\frac{1}{2}(x+y)\| \leq 1-\delta$ .

In other words,  $X$  is uniformly convex if for any two points  $x, y$  on the unit sphere  $S(0,1)$  the midpoint of the segment joining  $x$  to  $y$  can be close to but not on that sphere only

if  $x$  and  $y$  are sufficiently close to each other.

It is easily seen that any Hilbert space is uniformly convex. To show this, it suffices to recall that in a Hilbert space the equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

holds for any pair of vectors  $x, y$ . Hence it easily follows that  $\delta$  may be chosen equal to  $\varepsilon^2/8$ .

It is well known (see Clarkson [3]) that for  $1 < p < +\infty$  the spaces  $l^p$  of all infinite (real or complex) sequences  $(c_1, c_2, \dots)$  such that

$$\sum_{i=1}^{\infty} |c_i|^p < +\infty$$

are uniformly convex. For  $p = 1$  this is no longer true - in the space  $l^1$  the midpoint of the segment joining points  $(1, 0, 0, \dots)$ ,  $(0, 1, 0, \dots)$  of the unit sphere also lies on that sphere. Similarly, the example of points  $(1, 1, 0, 0, \dots)$ ,  $(0, 1, 1, 0, \dots)$  shows that neither the space  $l^\infty$  of all bounded sequences  $(c_1, c_2, \dots)$  with the norm equal to  $\sup \{|c_i| : i = 1, 2, \dots\}$ , nor the space  $c_0$  of all sequences in  $l^\infty$  such that  $c_i \rightarrow 0$  as  $i \rightarrow +\infty$ , with the same norm, are uniformly convex.

In a set  $S$  let be given a  $\sigma$ -algebra of subsets on which a nontrivial measure  $\mu$  is defined. Let  $L^p(\mu)$  be the

space of all  $\sigma$ -measurable functions  $x$  whose  $p$ -th powers are  $\sigma$ -integrable, provided with the usual norm

$$\|x\| = \left( \int_S |x|^p d\mu \right)^{1/p}.$$

It is known (see Clarkson [3]) that for  $1 < p < +\infty$  the space  $L^p(\mu)$  is uniformly convex. In particular, for the Lebesgue measure in a compact interval  $\Delta$  of the real line the space  $L^p(\Delta)$  of all Lebesgue measurable functions whose  $p$ -th powers are Lebesgue integrable is uniformly convex. And it may be easily seen that this fails to be true for  $p = 1$  as well as for the space  $C(\Delta)$  of all functions continuous in  $\Delta$ , provided with the norm of the uniform convergence.

It is worth mentioning that the uniform convexity is not a topological property, but rather a metrical one. For instance, the two-dimensional vector space  $R \times R$  is uniformly convex when endowed with the usual Euclidean norm, but fails to possess this property when endowed with the norm  $\|(c_1, c_2)\| = |c_1| + |c_2|$ , although these two norms are equivalent.

The following two lemmas are immediate consequences of Definition 1.1.

Lemma 1.1. If  $X$  is a uniformly convex Banach space, then, for any  $d > 0$  and  $\varepsilon > 0$ , the inequalities  $\|x\| \leq d$ ,  $\|y\| \leq d$ ,  $\|x-y\| \geq \varepsilon$  imply that

$$\left\| \frac{1}{2}(x+y) \right\| \leq (1-\delta(\varepsilon/d))d.$$

Proof. It is easily seen that by a proper dilation or contraction our Lemma reduces to the following statement: if  $\|x\| = 1$ ,  $\|y\| \leq 1$  and  $\|x-y\| \geq \varepsilon$ , then  $\|\frac{1}{2}(x+y)\| \leq 1-\delta(\varepsilon)$ .

We can choose points  $y_1, y_2$  on the unit sphere in such a manner that

$$y = \lambda_1 y_1 + \lambda_2 y_2 \quad (\lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1), \quad \|x-y_1\| \geq \varepsilon, \quad \|x-y_2\| \geq \varepsilon.$$

(The straight line through  $y_1$  and  $y_2$  should be a supporting straight line at  $y$  of the ball  $B(x, \|x-y\|)$ ; by the known properties of convex sets in Banach spaces, such a straight line always exists.) We have, therefore,

$$\|\frac{1}{2}(x+y)\| \leq \|\lambda_1(\frac{1}{2}x + \frac{1}{2}y_1)\| + \|\lambda_2(\frac{1}{2}x + \frac{1}{2}y_2)\| \leq \lambda_1(1-\delta) + \lambda_2(1-\delta) = 1-\delta,$$

and the proof is complete.

Lemma 1.2 (Schaefer [7]). Let  $X$  be a uniformly convex Banach space. Then for any  $\varepsilon > 0$ ,  $d > 0$  and  $\alpha \in (0, 1)$  the inequalities  $\|x\| \leq d$ ,  $\|y\| \leq d$  and  $\|x-y\| \geq \varepsilon$  imply that

$$\|\alpha x + \beta y\| \leq (1-2\delta(\varepsilon/d)\min(\alpha, \beta))d \quad (\beta = 1-\alpha).$$

Proof. Without loss of generality we may assume that



$0 < \alpha \leq \frac{1}{2}$ . Then

$$\begin{aligned}\|\alpha x + \beta y\| &= \|\alpha(x+y) + (\beta-\alpha)y\| \leq 2\alpha\|\frac{1}{2}(x+y)\| + (\beta-\alpha)\|y\| \\ &\leq 2\alpha(1-\delta(\varepsilon/d))d + (\beta-\alpha)d = (1-2\alpha\delta(\varepsilon/d))d\end{aligned}$$

and the proof is complete.

## 2. Strictly convex Banach spaces

Definition 1.2 (Clarkson [3]). A Banach space  $X$  is called strictly convex (rotund, in the terminology of [4]) if for any pair of vectors  $x, y$  in  $X$  from  $\|x+y\| = \|x\| + \|y\|$  it follows that  $x = \lambda y$ ,  $\lambda > 0$  (or, in a trivial case,  $y = 0$ ).

It may be easily shown that none of spaces  $l^1, L^1, l^\infty, c_0$  and  $C(\Delta)$  is strictly convex. However, we have the following general:

Proposition 1.1 (Clarkson [3]). Every uniformly convex Banach space is strictly convex.

Proof. Suppose that  $0 < \|y\| < \|x\|$  and  $\|x+y\| = \|x\| + \|y\|$ . Setting  $\lambda = \|x\|/\|y\|$ , we have

$$\lambda(\|x\| + \|y\|) = \lambda\|x+y\| \leq \|x+\lambda y\| + \|(\lambda-1)x\| \leq \lambda(\|x\| + \|y\|).$$

Hence,  $\|x+\lambda y\| = \|x\| + \lambda\|y\| = 2\|x\|$ . On the other hand, by the uniform convexity of  $X$ , if  $x \neq \lambda y$ , then  $\frac{1}{2}\|x+\lambda y\| < \|x\|$  which

yields a contradiction and completes the proof.

3. Convex sets with normal structure

Let  $C$  be a convex bounded set in a Banach space  $X$ , of diameter  $d$ . A point  $x$  in  $X$  is said to be diametral for  $C$  if  $\sup \{\|x-y\| : y \in C\} = d$ .

It is easily seen, for instance, that in the Banach space  $C[0,1]$  every point of the convex and bounded set  $\{x(t) : 0 \leq x(t) \leq 1, x(0) = 0, x(1) = 1\}$  is diametral.

Definition 1.3 (Brodskii and Mil'man [1]). A convex set  $K$  in a Banach space  $X$  is said to have normal structure if for each bounded convex subset  $C$  of  $K$  which contains more than one point there exists a point  $x$  in  $C$  which is not diametral for  $C$ .

Geometrically,  $K$  has normal structure if for every bounded and convex subset  $C$  of  $K$  there exists a ball of radius less than the diameter of  $C$  centered at a point of  $C$  and containing  $C$ .

The following proposition gives a large class of sets with normal structure:

Proposition 1.2 (Brodskii and Mil'man [1]). Every convex and compact set of a Banach space has normal structure.

Proof. Suppose that a compact and convex set  $K$  of a Banach space  $X$  does not have normal structure. Let  $d > 0$  be

the diameter of  $K$ . Without loss of generality we may assume that all points of  $K$  are diametral for  $K$ . We shall construct a sequence  $x_1, x_2, \dots$  of points of  $K$  such that

$$(1.1) \quad \|x_i - x_k\| = d \quad (i, k = 1, 2, \dots; i \neq k)$$

which will yield a contradiction with the compactness of  $K$ . To this end we choose an arbitrary point  $x_1$  in  $K$  and assume that points  $x_1, \dots, x_n$  have been already chosen and satisfy (1.1). Since  $(x_1 + \dots + x_n)/n$  is a diametral point of  $K$  and  $K$  is compact, we can find in  $K$  a point  $x_{n+1}$  such that

$$\|x_{n+1} - \frac{x_1 + \dots + x_n}{n}\| = d.$$

Hence

$$d = \left\| \frac{(x_{n+1} - x_1) + \dots + (x_{n+1} - x_n)}{n} \right\| \leq \frac{1}{n} (\|x_{n+1} - x_1\| + \dots + \|x_{n+1} - x_n\|) = d$$

and therefore  $\|x_{n+1} - x_i\| = d$  for  $i = 1, \dots, n$ .

It is obvious that if a convex set  $K$  has normal structure, then so does every convex subset of  $K$ . In particular, if the whole space  $X$  has normal structure, then so does any convex set in  $X$ .

The above-mentioned example shows that the space  $C[0,1]$  does not have normal structure. It is also easy to show that in

the space  $c_0$  the convex bounded set  $\{(c_1, c_2, \dots): 0 \leq c_i \leq 1, i = 1, 2, \dots\}$  of the unit ball does not have normal structure. Similar examples show that the spaces  $l^1$  and  $L^1$  do not have normal structure either.

The following proposition exhibits a large class of spaces with normal structure.

Proposition 1.3 (Edelstein [5], Browder [2]). Every uniformly convex Banach space  $X$  has normal structure.

Proof. Let  $C$  be a bounded convex set in  $X$  containing at least two different points  $x_1, x_2$ . Let  $d$  be the diameter of  $C$  and let  $x_0 = \frac{1}{2}(x_1 + x_2)$ . For any  $x$  in  $C$  we have  $\|x - x_1\| \leq d, \|x - x_2\| \leq d$  so that, by Lemma 1.1,

$$\|x - x_0\| \leq (1 - \delta(\|x_1 - x_2\|/d))d,$$

which means that  $C$  is contained in the ball of radius less than  $d$  centered at  $x_0$ .

#### 4. Dual spaces and weak topology

For a given Banach space  $X$ ,  $X^*$  will denote its first conjugate (dual) space, i.e. the linear space of all linear continuous functionals  $u: X \rightarrow \mathbb{R}$  (or  $\mathbb{C}$ , if  $X$  is a complex Banach space), endowed with the usual norm (we denote by  $(u, x)$  the value  $u(x)$  of  $u$  at  $x$ ):

$$\|u\| = \sup \{ |(u, x)| : \|x\| \leq 1 \}.$$

Using  $X^*$  we introduce the weak topology in  $X$  in the following way. For a given  $\varepsilon > 0$  and a finite number of elements  $u_1, \dots, u_n$  of  $X^*$ , let

$$V(u_1, \dots, u_n; \varepsilon) = \{x \in X: |(u_i, x)| < \varepsilon, \quad i = 1, \dots, n\}.$$

We denote by  $\mathcal{V}$  the family of all sets  $V(u_1, \dots, u_n; \varepsilon)$  for any choice of  $\varepsilon$  and any finite sequence  $u_1, \dots, u_n$ . It may be easily verified that  $\mathcal{V}$  satisfies all assumptions of the definition of a basis of neighborhoods of zero in a linear space. Thus, the following definition makes sense:

Definition 1.4. A topology defined by the basis  $\mathcal{V}$  of neighborhoods of zero in  $X$  is called the weak topology of  $X$ .

It is easily seen that in this topology, which is obviously coarser than the usual norm topology of  $X$ , a sequence  $\{x_n\} \subset X$  converges weakly to  $x_0$  in  $X$  if and only if

$$\lim_{n \rightarrow \infty} (u, x_n) = (u, x_0)$$

for any  $u$  in  $X^*$ . Every weakly convergent sequence  $\{x_n\}$  is necessarily bounded; moreover, the norm of its limit is less than or equal to  $\liminf_{n \rightarrow \infty} \|x_n\|$ .

The space  $X$  endowed with its weak topology is a linear locally convex topological space. In the sequel by the terms weakly closed set, weakly compact set, weak closure of a set

etc. we shall mean closed or compact set, closure of a set etc. in the weak topology. All usual topological terms will refer to the norm topology of  $X$ , sometimes called the strong topology of  $X$ .

The norm topology of a Banach space  $X$  and its weak topology are equivalent if and only if  $X$  is of finite dimension.

In a Hilbert space  $X$  with the scalar product  $(\cdot, \cdot)$ , for any fixed vector  $y$  in  $X$  the formula

$$(1.2) \quad u(x) = (x, y)$$

defines an element  $u$  of  $X^*$ , and conversely, for every  $u$  in  $X^*$  there exists a uniquely determined element  $y \in X$  such that (1.2) holds true. Moreover, the norm  $\|u\|$  of  $u$  is then equal to  $\|y\|$ . For this reason  $X$  and  $X^*$  are usually completely identified with each other.

For  $1 < p < +\infty$ , in the space  $l^p$  the general form of a linear continuous functional  $u$  is given by an explicit formula,

$$(1.3) \quad u[(c_1, c_2, \dots)] = \sum_{i=1}^{\infty} d_i c_i$$

with  $(d_1, d_2, \dots)$  in  $l^q$  where  $1/p + 1/q = 1$ . Moreover,  $\|u\|$  is equal to the norm of  $(d_1, d_2, \dots)$  in  $l^q$ . This allows one to identify the dual space  $(l^p)^*$  with  $l^q$ .

For a given measure  $\mu$  on a  $\sigma$ -algebra of subsets of a

set  $S$ , the general form of a linear continuous functional  $u$  on the space  $L^p(\mu)$  ( $1 < p < +\infty$ ) is given by the formula

$$(1.4) \quad u(x) = \int_S xy d\mu$$

where  $y$  is an element of the space  $L^q(\mu)$  ( $1/p + 1/q = 1$ ) and  $\|u\|$  is equal to the norm of  $y$  in  $L^q(\mu)$ . For this reason the dual space  $(L^p(\mu))^*$  is usually identified with the space  $L^q(\mu)$ .

In the space  $l^1$  every linear continuous functional  $u$  is of the form (1.3) with  $(d_1, d_2, \dots)$  in the space  $l^\infty$  and the corresponding norms are equal, so that the space  $(l^1)^*$  may be identified with the space  $l^\infty$ .

Similarly, in the space  $L^1(\mu)$  any linear continuous functional  $u$  is of the norm (1.4) with  $y$  essentially bounded (i.e. bounded except possibly on a subset of  $S$  of measure zero) in  $S$ . For this reason the dual space  $(L^1(\mu))^*$  is identified with the space  $L^\infty(\mu)$  of all essentially bounded  $\sigma$ -measurable functions  $y$  on  $S$  with the norm equal to the "essential supremum" of  $|y|$ .

By the classical Riesz theorem, every linear continuous functional  $u$  in the space  $C(\Delta)$  is of the form

$$u(x(t)) = \int_{\Delta} x(t) dy(t)$$

where  $y(t)$  is a function of bounded variation on  $\Delta$ . The norm

$\|u\|$  is equal to the total variation of  $y(t)$  on  $\Delta$ .

In the sequel we shall need the following simple property of weakly convergent sequences in a Hilbert space.

Lemma 1.3. If in a Hilbert space  $X$  the sequence  $\{x_n\}$  is weakly convergent to  $x$ , then for any  $y \neq x$ :

$$(1.5) \quad \liminf_{n \rightarrow \infty} \|x_n - y\| > \liminf_{n \rightarrow \infty} \|x_n - x\|.$$

Proof. Since every weakly convergent sequence is necessarily bounded, both limits in (1.5) are finite. Thus, to prove (1.5), it suffices to observe that in the relationship

$$\|x_n - y\|^2 = \|x_n - x + x - y\|^2 = \|x_n - x\|^2 + \|x - y\|^2 + 2\operatorname{Re} (x_n - x, x - y)$$

the last term goes to zero as  $n$  goes to infinity.

The following theorem states one of the fundamental results of the geometric theory of Banach spaces.

Theorem 1.1 (Mazur). Each closed convex set of a Banach space is necessarily weakly closed.

Let  $C$  be a set in a Banach space  $X$ . The closure of the set

$$\{\lambda_1 x_1 + \dots + \lambda_k x_k : \lambda_1, \dots, \lambda_k \geq 0, \lambda_1 + \dots + \lambda_k = 1; x_1, \dots, x_k \in C\}$$



is called the convex closure (convex hull) of  $C$  and is denoted by  $\text{ccl } C$ . It is easily seen that equivalently  $\text{ccl } C$  may be defined as the smallest closed convex set in  $X$  which contains  $C$ . In other words, an element  $x$  in  $X$  belongs to the convex closure of  $C$  if and only if for any  $\varepsilon > 0$  there exist a finite sequence of vectors  $x_1, \dots, x_k$  in  $C$  and a sequence  $\lambda_1, \dots, \lambda_k$  of nonnegative real numbers such that

$$\|\lambda_1 x_1 + \dots + \lambda_k x_k - x\| \leq \varepsilon \quad (\lambda_1 + \dots + \lambda_k = 1).$$

The following statement is a simple consequence of Theorem 1.1.

Theorem 1.2. The weak closure of every bounded set of a Banach space is contained in its convex closure.

Equivalently, if the sequence  $\{x_n\}$  converges weakly to  $x$ , then for every  $\varepsilon > 0$  and any positive integer  $m$  there is a finite sequence  $\lambda_1, \dots, \lambda_k$  of nonnegative real numbers such that

$$\|\lambda_1 x_{m+1} + \dots + \lambda_k x_{m+k} - x\| \leq \varepsilon \quad (\lambda_1 + \dots + \lambda_k = 1).$$

As a simple illustration of Theorem 1.2., consider in a given compact interval  $\Delta$  of the real line the sequence of functions  $\{\sin nt\}$ . By the classical Riemann-Lebesgue theorem, for every function  $x(t)$  integrable in  $\Delta$  we have

$$\lim_{n \rightarrow \infty} \int_{\Delta} x(t) \sin nt dt = 0.$$

In other words, for every linear functional  $u$  in the space  $L^p(\Delta)$  ( $1 < p < +\infty$ ) we have

$$\lim_{n \rightarrow \infty} u(\sin nt) = 0 = u(0),$$

i.e. the sequence  $\{\sin nt\}$  is weakly convergent in  $L^p(\mu)$  to zero. It is easy to verify that the sequence

$$\frac{1}{n}(\sin t + \dots + \sin nt) = \frac{1}{2n} \left[ \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} - 1 \right]$$

converges to zero in the norm topology of the space  $L^p(\mu)$ .

In the dual space  $X^*$  of a Banach space  $X$  the family  $\mathcal{V}^*$  of sets  $V^*(x_1, \dots, x_n; \varepsilon) = \{u \in X^* : |(u, x_i)| < \varepsilon, i = 1, \dots, n\}$  ( $\varepsilon > 0; x_1, \dots, x_n \in X$ ) defines a basis of neighborhoods of zero of a topology which is called the weak\* topology in  $X^*$ . A sequence  $\{u_n\} \subset X^*$  converges weakly\* to  $u_0$  in  $X^*$  if and only if

$$\lim_{n \rightarrow \infty} (u_n, x) = (u_0, x)$$

for any  $x$  in  $X$ .

## 5. Reflexive Banach spaces

For any fixed vector  $x$  in a Banach space  $X$ , the mapping of  $X^*$  into  $\mathbb{R}$  (or  $\mathbb{C}$ , if  $X$  is a complex Banach space) which to every  $u$  in  $X^*$  assigns the value  $(u, x)$  of  $u$  at  $x$  is a linear continuous functional in the space  $X^*$ , i.e. an element of the space  $(X^*)^*$  noted also as  $X^{**}$ . Moreover, the norm of this functional is equal to the norm  $\|x\|$ . It may be easily verified that the canonical mapping of  $X$  into  $X^{**}$  defined by this correspondence between elements of  $X$  and linear continuous functionals on  $X^*$  is linear and one-to-one. Therefore, it is an isometrical imbedding of  $X$  into  $X^{**}$ .

Definition 1.5. A Banach space  $X$  is called reflexive if the canonical imbedding of  $X$  into  $X^{**}$  is onto.

It is clear that every Hilbert space is reflexive. For  $1 < p < +\infty$  the spaces  $l^p$  and  $L^p(\mu)$  are reflexive which follows immediately from the general forms of linear continuous functionals in those spaces. The space  $c_0$  is not reflexive. Its dual space  $c_0^*$  is isometric to the space  $l^1$  and, in turn, the dual space of the latter is isometric to the space  $l^\infty$  which is essentially larger than  $c_0$ .

The following theorem exhibits a large class of reflexive Banach spaces.

Theorem 1.3 (Mil' man, Pettis). Every uniformly convex Banach space is reflexive.

The fundamental property of reflexive Banach spaces is stated in the following:

Theorem 1.4 (Bourbaki, Kakutani). A Banach space  $X$  is reflexive if and only if its unit ball is weakly compact.

In other words, a Banach space  $X$  is reflexive if and only if it is locally (in the sense of the norm topology) weakly compact.

From Theorems 1.1 and 1.4 it follows immediately that in a reflexive Banach space every bounded closed and convex set is weakly compact.

In a somewhat different way Theorem 1.4 may be stated in the form of the following:

Theorem 1.5 (Smulyan, Eberlein). A Banach space  $X$  is reflexive if and only if every bounded sequence of elements of  $X$  contains a subsequence which is weakly convergent.

Theorems 1.4 and 1.5 look as though they were identical but we have to notice that the weak\* topology, in general, does not satisfy any axiom of countability and therefore the weak compactness is not necessarily equivalent to the weak sequential compactness.

In general, the weak topology in the dual space  $X^*$  of a Banach space  $X$  is finer than the weak\* topology in  $X^*$ . It is clear, however, that these two topologies coincide if the space  $X$

is reflexive.

6. Hilbert space structure in finite dimensional linear spaces

Let  $X$  be a (real, for the sake of simplicity)  $n$ -dimensional linear space and  $E = (e_1, \dots, e_n)$  a given basis in  $X$ . Setting, for  $x = \alpha_1 e_1 + \dots + \alpha_n e_n$  and  $u = \lambda_1 e_1 + \dots + \lambda_n e_n$ ,

$$\langle u, x \rangle = \lambda_1 \alpha_1 + \dots + \lambda_n \alpha_n,$$

we define in  $X$  a scalar product and introduce in  $X$  the structure of a finite dimensional Hilbert (Euclidean) space which, as may be easily verified, is topologically equivalent to any Banach space structure in  $X$  (actually, all Banach space structures in a finite dimensional linear space are equivalent to each other).

The dual space  $X^*$  of the space  $X$  is defined in a purely algebraic manner as a linear space of all linear mappings of  $X$  into  $R$  and has the same dimension as  $X$ . Introducing in  $X^*$  the dual basis  $E^* = (e_1^*, \dots, e_n^*)$  defined by the conditions

$$(e_i^*, e_j) = \delta_{ij} \quad (i, j = 1, \dots, n),$$

we can identify every element  $u = \lambda_1 e_1^* + \dots + \lambda_n e_n^*$  of  $X^*$  with the corresponding element  $u = \lambda_1 e_1 + \dots + \lambda_n e_n$  of  $X$ . Under this identification, for each  $x$  in  $X$ , we have

$$(u, x) = \langle u, x \rangle.$$

Therefore, we can always consider a finite dimensional linear space  $X$  as a Euclidean space, treat its dual space  $X^*$  as identical to  $X$  and, the bilinear form  $(u, x)$  as equal to the scalar product in  $X$ .

## 7. Adjoint mappings

Let  $L$  be a linear mapping defined in a linear subspace  $D(L)$  of a Banach space  $X$  with values in the dual space  $X^*$ .  $L$  is said to be densely defined if the subspace  $D(L)$  is dense in  $X$ .

For a given densely defined linear mapping  $L: D(L) \rightarrow X^*$  and each given  $x$  in  $X$ , the formula

$$L_x y = (Ly, x) \quad \text{for all } y \text{ in } D(L)$$

defines a linear mapping  $L_x: D(L) \rightarrow R$ . The set  $D(L^*) \subset X$  of all elements  $x$  of  $X$  for which  $L_x$  is a bounded mapping, is a linear subspace of  $X$ . By the Hahn-Banach theorem, for every  $x$  in  $D(L^*)$  there exists in  $X^*$  a uniquely determined element  $L^*x$  such that  $L_x y = (L^*x, y)$  for all  $y$  in  $D(L)$ ; i.e.,

$$(1.6) \quad (Ly, x) = (L^*x, y)$$

for all  $y$  in  $D(L)$  and all  $x$  in  $D(L^*)$ .

It is easily seen that the mapping  $L^*: D(L^*) \rightarrow X^*$  defined for every  $x$  in  $D(L^*)$  by formula (1.6), is linear.  $L^*$  is

called the adjoint mapping of the mapping  $L$ .

A linear mapping  $L:D(L) \rightarrow X^*$  is said to be closed if its graph is a closed subset of  $X \times X^*$ . If  $L$  is not closed, it is said to be closeable if there exists a linear closed mapping  $L':D(L') \rightarrow X^*$  containing  $L$ , i.e. such that  $D(L) \subset D(L')$  and  $L = L'$  in  $D(L)$ . The closure of a closeable densely defined linear mapping  $L:D(L) \rightarrow X^*$  is the (uniquely determined) least closed linear mapping containing  $L$ .

A linear densely defined mapping  $L:D(L) \rightarrow X^*$  is closeable if and only if its adjoint mapping  $L^*$  is densely defined.

The adjoint mapping  $L^*$  of a densely defined linear mapping  $L$  is always closed. If a densely defined linear mapping  $L$  is closeable so that the adjoint mapping  $L^*$  is densely defined, we may form the second adjoint mapping  $L^{**} = (L^*)^*$ . If  $L$  is closed, then  $L^{**} = L$ .

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Nevertheless for nonexpansive mappings a quite general and useful theory of fixed points can be constructed. Its fundamental result is contained in the following:

Theorem 2.1 (Kirk[13]). If  $C$  is a convex closed and bounded set with normal structure of a reflexive Banach space  $X$ , then every nonexpansive mapping  $U: C \rightarrow C$  has a fixed point.

Simple examples show that Theorem 2.1 fails to be true without the assumption that  $C$  has normal structure and that the space  $X$  is reflexive. For instance, the mapping  $U: c_0 \rightarrow c_0$  defined above maps the unit ball in  $c_0$  into itself but, as we have seen, does not have fixed points. Similarly (Kirk [3]), the mapping  $U: C[0,1] \rightarrow C[0,1]$  which to every function  $x(t)$  continuous in  $[0,1]$  assigns the function  $tx(t)$  is nonexpansive and maps the convex and closed bounded set

$$C = \{x(t): 0 \leq x(t) \leq 1, x(0) = 0, x(1) = 1\}$$

into itself. However, the unique fixed point of  $U$ , the function  $x(t) = 0$ , does not belong to  $C$ .

The proof of Theorem 2.1 will consist of two parts. In the first, using the assumption of the reflexivity of the space  $X$  but without using the nonexpansivity of the mapping  $U$ , we shall prove the existence of a minimal closed and convex subset of  $C$  invariant under  $U$ . In the second, it will be shown that the normal structure of  $C$  and the nonexpansivity of  $U$  imply that

such a minimal invariant set cannot contain more than one element which, therefore, is necessarily a fixed point of  $U$ .

Denote by  $\Phi$  the family of all convex closed and non-empty subsets  $C'$  of  $C$  such that  $U(C') \subset C'$ . The family  $\Phi$  is nonempty since  $C$  belongs to it. In an obvious manner  $\Phi$  may be ordered (partially) by the relation of inclusion. It is easy to show that  $\Phi$  is inductive. To prove this, consider an ordered subfamily  $\Psi$  of  $\Phi$ . The intersection

$$C^* = \bigcap_{C' \in \Psi} C'$$

is a convex closed and invariant subset of  $C$ . All sets  $C'$  in  $\Psi$  are weakly closed (Theorem 1.1) and the family  $\Psi$  has finite intersection property. By weak compactness of  $C$  (Theorem 1.4), it follows that  $C^*$  is nonempty so that  $C^*$  belongs to  $\Phi$  and is a lower bound for  $\Psi$ .

Now, by the Kuratowski-Zorn lemma, there exists in  $\Phi$  a minimal element, say  $C_0$ . Observe first that  $C_0$  is equal to  $\text{ccl } U(C_0)$  since  $\text{ccl } U(C_0)$  is contained in  $C_0$ , closed convex and invariant.

Suppose that the diameter  $d$  of  $C_0$  is positive. Since  $C$  has normal structure, by Definition 1.3, there exists in  $C_0$  a point  $x_0$  such that  $C_0 \subset B(x_0, d_1)$  for some  $d_1 < d$ . Let

$$C_1 = \{x \in C_0 : C_0 \subset B(x, d_1)\} = C_0 \cap \left\{ \bigcap_{y \in C_0} B(y, d_1) \right\}.$$

$C_1$  is a convex and closed subset of  $C_0$ , nonempty since  $x_0 \in C_1$ . The inequality  $d_1 < d$  implies that  $C_1$  is different from  $C_0$ . We shall prove that  $C_1$  is invariant under  $U$ . Indeed, for  $x$  in  $C_1$  we have

$$\|Ux - Uy\| \leq \|x - y\| \leq d_1 \quad (y \in C_0).$$

Thus  $U(C_0) \subset B(Ux, d_1)$ . But then  $\text{ccl } U(C_0) \subset B(Ux, d_1)$  and finally  $C_0 \subset B(Ux, d_1)$  which means that  $Ux \in C_1$ . Summing up,  $C_1$  is an invariant closed convex subset of  $C_0$  which contradicts the minimality of  $C_0$ .

As an immediate consequence of Theorem 1.2 we have the following:

Theorem 2.2 (Browder [3]). If  $U: C \rightarrow C$  is a nonexpansive mapping of a convex closed and bounded set  $C$  in a uniformly convex Banach space  $X$  into itself, then  $U$  has a fixed point in  $C$ .

Proof. By Proposition 1.3,  $C$  has normal structure, and by Theorem 1.3,  $X$  is a reflexive space. Therefore, Theorem 2.2 follows immediately from Theorem 2.1.

Let us observe that if the image  $U(C)$  of the set  $C$  is compact, then Theorem 2.1 is a special case of the fixed point theorem of Schauder. Similarly, if the mapping  $U$  is weakly continuous, then it is a special case of the Tikhonov fixed point

theorem. But even very simple nonexpansive mappings may fail to possess these properties. For instance, for an infinite dimensional Banach space  $X$  the identity mapping of  $X$  into itself is not compact and the mapping  $x \rightarrow \|x\|x_0$ , where  $x_0$  is an arbitrary element of  $X$  with  $\|x_0\| = 1$ , is not weakly continuous since the norm is not a weakly continuous functional; it is clear that combining these two mappings one easily obtains a nonexpansive mapping of  $X \times X$  which is neither compact nor weakly continuous.

The following corollary which for isometrical mappings has been proved in [9] is a slight modification of Theorem 2.1.

Corollary 2.1 (Kirk[13]). If in Theorem 2.1 the condition that  $C$  be bounded is replaced by the requirement that the sequence  $S = \{U^n x\}$  be bounded for some  $x$  in  $C$ , then  $U$  has a fixed point.

Proof. Let  $S \subset B(x, r)$  for some  $r > 0$ . Then  $x$  belongs to  $B_n = B(U^n x, r)$  for  $n = 1, 2, \dots$  so that

$$D = \left( \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} B_k \right) \cap C$$

is a nonempty subset of  $C$ . Moreover,  $D$  is convex (as a union of an increasing sequence of convex sets), bounded and invariant under  $U$ . Therefore, its closure  $\bar{D}$  is a convex closed bounded set mapped by  $U$  into itself. Hence, applying Theorem 2.1, we conclude that  $U$  has a fixed point in  $D$ .

The following example (Edelstein [10]) shows that Corollary 2.1 would no longer be true if we replaced the sequence  $S$  by a subsequence. In the space  $l^2$  the mapping  $U: \{c_k\} \rightarrow \{e^{2\pi i/k!}(c_k-1)+1\}$  does not have fixed points although, as it may be easily verified, the sequence  $\{U^n 0\}$  converges to zero.

For strictly convex Banach spaces we have, in addition, the following useful information on the set of fixed points of a nonexpansive mapping.

Proposition 2.1. If  $C$  is a convex set of a strictly convex Banach space  $X$ , then for every nonexpansive mapping  $U: C \rightarrow X$  the set

$$F_U = \{x \in C: Ux = x\}$$

is convex and relatively closed in  $C$ . In particular, if  $C$  is closed, then  $F_U$  is also closed.

Proof. If  $x_1, x_2 \in F_U$ , then for any  $x$  on the segment joining  $x_1$  and  $x_2$  we have

$$\begin{aligned} \|x_1 - x_2\| &\leq \|x_1 - Ux\| + \|Ux - x_2\| = \|Ux_1 - Ux\| + \|Ux - Ux_2\| \leq \|x_1 - x\| + \\ &\quad + \|x - x_2\| = \|x_1 - x_2\| \end{aligned}$$

and therefore

$$\|x_1 - x_2\| = \|x_1 - Ux\| + \|Ux - x_2\|, \quad \|x_1 - x\| = \|x_1 - Ux\|.$$

Hence, by the strict convexity,  $Ux$  lies on the segment joining  $x_1$  and  $x_2$ , and therefore  $x = Ux$ . The closedness of  $F_U$  follows immediately from the continuity of  $U$ .

Proposition 2.1 fails to be true without the assumption of strict convexity as shown by the following example (DeMarr [8]). On the plane  $R^2$  with the norm  $\|(a,b)\| = \max\{|a|, |b|\}$  the mapping  $U:(a,b) \rightarrow (|b|, b)$  is nonexpansive and  $(1,1), (1,-1)$  are fixed points of  $U$  while  $(1,0)$  is not a fixed point.

## 2. Isometric mappings

A mapping  $U:C \rightarrow X$  of a set  $C$  in a Banach space  $X$  into  $X$  is called isometric or an isometry if

$$\|Ux - Uy\| = \|x - y\|$$

for all  $x$  and  $y$  in  $C$ .

Since every isometry is a nonexpansive mapping, Theorems 2.1 and 2.2 apply to isometries. It turns out, however, that for isometric mappings more precise information about their fixed points is available. Namely, under rather general assumptions on the set  $C$  there exists a uniquely determined point in  $C$  which is a common fixed point for all isometries of  $C$  into itself. This is the so-called center of  $C$  which, following Brodskii and Mil'man [2], can be constructed by transfinite induction in the following manner.

Let  $C$  be a closed convex and bounded subset of a

Banach space  $X$ . Assume furthermore that  $C$  is weakly compact (it is so, in particular, if  $X$  is reflexive (by Theorem 1.4) or if  $C$  is compact). Finally, suppose that  $C$  has normal structure. Note that, by Proposition 1.2, the latter assumption automatically is satisfied if  $C$  is compact.

To begin with, we put  $C_1 = C$ . Let  $\alpha$  be an ordinal number and, for all ordinal numbers  $\beta < \alpha$ , assume that nonempty closed and convex (and therefore weakly closed) sets  $C_\beta$  have already been constructed and that  $C_\beta$  is a proper subset of  $C_\gamma$  whenever  $\gamma < \beta$ . The nonempty (by the weak compactness of  $C$ ) intersection

$$D = \bigcap_{\beta < \alpha} C_\beta$$

is a closed convex subset of  $C$ . If  $D$  has only one element, we put  $C_\gamma = D$  for all ordinal numbers  $\gamma \geq \alpha$ . If not, let  $d$  be the diameter of  $D$ . For any positive  $\delta \leq d$ , the set

$$D(\delta) = \{x \in D : D \subset B(x, \delta)\} = D \cap \left\{ \bigcap_{y \in D} B(y, \delta) \right\}$$

is closed and convex, and therefore weakly closed; moreover,  $\delta' < \delta$  implies that  $D(\delta') \subset D(\delta)$ . From the weak compactness of  $D$  (which is a weakly closed subset of a weakly compact set  $C$  and therefore is itself weakly compact) it follows that there is  $\delta_0 > 0$  such that  $D(\delta_0) \neq \emptyset$  and  $D(\delta) = \emptyset$  for every  $\delta < \delta_0$ .

Now we put  $C_\alpha = D(\delta_\alpha)$ . Since  $D$  has normal structure,  $C_\alpha$  is a proper closed convex subset of  $D$ .

It is clear that the above construction defines a uniquely determined transfinite sequence

$$(2.1) \quad C_1, C_2, \dots, C_\omega, C_{\omega+1}, \dots$$

By the Kuratowski-Zorn lemma, from the properties of the elements of (2.1) it follows that, beginning with some ordinal number  $\alpha$ , all elements of (2.1) must be equal which is possible if and only if  $C_\alpha$  has exactly one element. This property determines a point of  $C$  called the center of this set.

It should be observed that if  $X$  is a uniformly convex Banach space, then the above construction reduces to one step only, since already  $C_2$  cannot contain more than one element (see N. A. Routledge [20] and V. L. Klee [15]). Indeed, if  $C_2 = D(\delta_0) = C_1(\delta_0)$  contained two distinct points  $x_1$  and  $x_2$ , then by the argument already used in the proof of Proposition 1.3 it would be easy to show that  $C_1$  is contained in a ball centered at  $x_0 = \frac{1}{2}(x_1 + x_2)$  of a radius less than  $\delta_0$  - a contradiction with the definition of  $C_2$ .

Theorem 2.3 (Brodskii and Mil'man [2]). The center of a closed convex bounded and weakly compact subset  $C$  with normal structure of a Banach space  $X$  is a common fixed point of all iso-



metric mappings of  $C$  into itself.

Proof. It is clear that if in the above construction of the set  $C_\alpha$  all sets  $C_\beta$  (for  $\beta < \alpha$ ) are invariant under isometries of  $C$ , then so is the set  $D$ . Now, it is easily seen that the sets  $D(\delta)$  are also invariant. Indeed, if  $Ux \notin D(\delta)$  for some isometry  $U: D \rightarrow D$ , then there is  $y$  in  $D$  such that  $\|Ux - y\| > \delta$ , and therefore  $\|x - U^{-1}y\| > \delta$  which implies that  $x \notin D(\delta)$ . In particular,  $C_\alpha$  is invariant. Since  $C_1$  is invariant, by transfinite induction all elements of sequence (2.1) are invariant sets under all isometries of  $C$  into itself and this clearly implies that the center of  $C$  has the same property.

### 3. Common fixed points of commuting nonexpansive mappings

A family  $\{U_\lambda\}_{\lambda \in \Lambda}$  of mappings of a set  $A$  into itself is called commutative or Abelian if  $U_\lambda U_\mu = U_\mu U_\lambda$  for all  $\lambda, \mu$  in  $\Lambda$ .

The well-known theorem of Markov [17] and Kakutani [2] states that if  $\{L_\lambda\}_{\lambda \in \Lambda}$  is a commutative family of linear continuous mappings of a compact set  $C$  of a linear locally convex topological space  $X$  into itself, then there exists in  $C$  a point  $x_0$  such that  $L_\lambda x_0 = x_0$  for all  $\lambda \in \Lambda$ . A similar result is valid for nonexpansive mappings. Namely, we have the following (for further generalizations see [1]):

Theorem 2.4 (Browder [4]). Let  $C$  be a bounded closed subset with normal structure of a reflexive and strictly convex Banach space  $X$ . If  $\{U_\lambda\}_{\lambda \in \Lambda}$  is a commutative family of non-expansive mappings of  $C$  into itself, then  $\{U_\lambda\}$  has a common fixed point in  $C$ . In particular, every commutative family of nonexpansive mappings of a bounded closed convex subset of a uniformly convex Banach space into itself has a common fixed point.

Proof. By Theorem 2.1 and Proposition 2.1, for every  $\lambda \in \Lambda$  the set  $F_\lambda$  of fixed points of the mapping  $U_\lambda$  is nonempty closed and convex, and therefore weakly closed. To prove that the intersection of all  $F_\lambda$  is nonempty, it suffices to prove that the family  $\{F_\lambda\}_{\lambda \in \Lambda}$  has finite intersection property.

Observe first that if  $x \in F_\lambda$  for some  $\lambda$  in  $\Lambda$ , then for any  $\mu \in \Lambda$  we have

$$U_\lambda(U_\mu x) = U_\mu(U_\lambda x) = U_\mu x$$

which means that  $U_\mu$  maps  $F_\lambda$  into itself. Now, to prove the finite intersection property by induction with respect to the number of sets, assume that for a given sequence  $\lambda_1, \dots, \lambda_m$  from  $\Lambda$  the intersection

$$F = F_{\lambda_1} \cap \dots \cap F_{\lambda_{m-1}}$$

is nonempty. We can consider  $U_{\lambda_m}$  as a nonexpansive mapping of  $F$

into itself and, by Theorem 2.1, conclude that the set  $F \cap F_{\lambda_m}$  is also empty.

The following result generalizes the Markov-Kakutani theorem in another direction.

Theorem 2.5 (DeMarr [8]). Every commutative family  $\{U_\lambda\}_{\lambda \in \Lambda}$  of nonexpansive mappings of a compact subset  $C$  of a Banach space  $X$  into itself has a common fixed point in  $C$ .

Proof. Using the strong compactness of  $C$  in place of the weak one, we can prove as in the proof of Theorem 2.1 that there exists a minimal closed convex subset  $C_0$  of  $C$  which is invariant under all mappings of the family  $\{U_\lambda\}$ . If  $C_0$  has only one element, then this element is clearly a common fixed point for  $\{U_\lambda\}$  and the proof is complete. If  $C_0$  has more than one element, then by a similar argument we can prove that there exists in  $C_0$  a minimal compact (not necessarily convex) subset  $K$  which is invariant under all mappings from  $\{U_\lambda\}$ .

If  $K$  has only one element, then the proof is complete. Assume, therefore, that the diameter  $d$  of  $K$  is positive. For  $\lambda$  in  $\Lambda$ ,  $U_\lambda(K)$  is a nonempty compact subset of  $K$ , invariant under all mappings from  $\{U_\lambda\}$  since for any  $\mu$  in  $\Lambda$  we have

$$U_\mu(U_\lambda(K)) = U_\lambda(U_\mu(K)) \subset U_\lambda(K).$$

Hence, by the minimality of  $K$ ,  $U_\lambda(K) = K$  for every  $\lambda$  in  $\Lambda$ .

The diameter of the convex closure of  $K$  is also equal to  $d$ . By Proposition 1.2, therefore, there exists  $d_1 < d$  such that  $K \subset B(x_0, d_1)$  for some  $x_0$  in  $C_0$ . Therefore,

$$C_1 = \{x \in C_0 : K \subset B(x, d_1)\} = C_0 \cap \left\{ \bigcap_{y \in C_0} B(y, d_1) \right\}$$

is a nonempty closed convex subset of  $C_0$ . The inequality  $d_1 < d$  implies that  $C_1$  is a proper subset of  $C_0$ . Furthermore, for any  $\lambda$  in  $\Lambda$ , by the nonexpansivity of  $U_\lambda$ , we have  $U_\lambda(K) \subset B(Ux, d_1)$  for every  $x$  in  $C_1$ . Therefore,  $K \subset B(Ux, d_1)$  so that  $Ux$  is also in  $C_1$ . This means that  $C_1$  is invariant under all mappings from  $\{U_\lambda\}$  which yields a contradiction with the minimality of  $C_0$  and completes the proof.

#### 4. Nonexpansive mappings and successive approximations

Trivial examples show that even in very simple cases the sequence of successive approximations for a nonexpansive mapping  $U$ , unlike for contractive mappings, may fail to be convergent. It suffices, for instance, to take for  $U$  a rotation in the plane around the origin of coordinates or a symmetry with respect to an arbitrary straight line. However, as pointed out by Krasnosel'skiĭ [16], in both examples one gets a convergent sequence of successive approximations if instead of  $U$  one takes the auxiliary non-expansive mapping  $\frac{1}{2}(I+U)$ , where  $I$  denotes the identical transformation of the plane; i.e., if the sequence  $\{x_n\}$  of successive

approximations is defined not by the usual recursive formula

$$(2.2) \quad x_{n+1} = Ux_n \quad (n = 0, 1, \dots)$$

but by the following one:

$$(2.3) \quad x_{n+1} = \frac{1}{2}(x_n + Ux_n) \quad (n = 0, 1, \dots).$$

The mappings  $U$  and  $\frac{1}{2}(I+U)$  have the same set of fixed points, so that the limit of a convergent sequence defined by (2.3) is necessarily a fixed point of  $U$ .

More generally, if  $C$  is a convex set in a Banach space  $X$  and the mapping  $U: C \rightarrow C$  is nonexpansive, then for any  $\alpha \in (0, 1)$  the mapping

$$(2.4) \quad U_\alpha = \alpha I + (1-\alpha)U$$

is nonexpansive and has the same fixed points as  $U$ . Therefore, the limit of a convergent sequence of successive approximations for  $U_\alpha$ , i.e. of a sequence  $\{x_n\}$  defined by the formula

$$(2.5) \quad x_{n+1} = \alpha x_n + (1-\alpha)Ux_n \quad (n = 0, 1, \dots),$$

is necessarily a fixed point of  $U$ .

Unlike for contractive mappings, it may happen that a

nonexpansive mapping has more than one fixed point. In this case, it turns out, the limit of a convergent sequence (2.5) can depend on the choice of the initial point  $x_0$  and on  $\alpha$  as well (see [21]).

If the mapping  $U:C \rightarrow C$  is nonexpansive, then for any positive integer  $n$  and any  $x$  in  $C$  we have

$$\|U^{n+1}x - U^n x\| = \|U(U^n x) - U(U^{n-1}x)\| \leq \|U^n x - U^{n-1}x\|$$

which means that the sequence  $\{\|U^{n+1}x - U^n x\|\}$  is nonincreasing.

Definition 2.2 (Browder and Petryshyn [6]). A nonexpansive mapping  $U:C \rightarrow C$  of a subset  $C$  of a Banach space  $X$  into itself is called asymptotically regular if

$$\lim_{n \rightarrow \infty} \|U^{n+1}x - U^n x\| = 0$$

for any  $x$  in  $C$ .

In other words,  $U:C \rightarrow C$  is asymptotically regular if for any  $x_0$  in  $C$  the sequence  $\{x_n\}$  of successive approximations defined by (2.2) is such that  $\|x_{n+1} - x_n\| = \|Ux_n - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

If  $C$  is a convex set, then for a given nonexpansive mapping  $U:C \rightarrow C$  and a given  $\alpha \in (0,1)$  we can consider the mapping  $U_\alpha$  defined by (2.4). Taking an  $x_0$  in  $C$ , we can form the sequence  $\{x_n\} = \{U_\alpha^n x_0\}$  defined by (2.5). Since

$$\|x_n - U_\alpha x_n\| = \|x_n - \alpha x_n - (1-\alpha)Ux_n\| = (1-\alpha)\|x_n - Ux_n\|,$$

the mapping  $U_\alpha$  is asymptotically regular if and only if

$$\lim_{n \rightarrow \infty} \|x_n - Ux_n\| = 0$$

for any  $x$  in  $C$ .

The concept of asymptotical regularity enables us to state in a very simple form some basic properties of nonexpansive mappings in uniformly convex Banach spaces.

Theorem 2.6 (Krasnosel'skiĭ [16] (for  $\alpha = \frac{1}{2}$ ), Schaefer [21]).

Let  $C$  be a convex set in a uniformly convex Banach space  $X$ .

Suppose that the mapping  $U: C \rightarrow C$  is nonexpansive and that the set

$$F = \{x \in C: Ux = x\}$$

of fixed points of  $U$  is nonempty. Then, for each  $\alpha \in (0,1)$ , the mapping  $U_\alpha$  is asymptotically regular.

Proof. Let  $x_0$  be a given element of  $C$  and let  $\{x_n\}$  be the sequence defined by (2.5). Since  $U_\alpha$  is a nonexpansive mapping, by the preceding remarks we have only to show that the non-increasing sequence  $\{\|x_n - Ux_n\|\}$  goes to zero as  $n$  goes to infinity.

Suppose to the contrary that  $\|x_n - Ux_n\| \geq \varepsilon > 0$  ( $n = 0,1,\dots$ )

and let  $y$  be an element of  $F$ . By the nonexpansivity of  $U$  we have

$$(2.6) \quad \|y - Ux_n\| = \|Uy - Ux_n\| \leq \|y - x_n\|$$

so that, by Lemma 1.2, there exists a  $\delta > 0$  such that

$$\begin{aligned} \|y - x_{n+1}\| &= \|y - (\alpha x_n + (1-\alpha)Ux_n)\| = \|\alpha(y - x_n) + (1-\alpha)(y - Ux_n)\| \\ &\leq (1-\alpha)\|y - x_n\|. \end{aligned}$$

Hence  $\|y - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$  and by (2.6) also  $\|y - Ux_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . This implies that  $\|x_n - Ux_n\| \rightarrow 0$  and this contradiction completes the proof.

Combining the preceding theorem with Theorem 2.2 we obtain immediately the following:

Corollary 2.2. If  $U$  is a nonexpansive mapping of a bounded closed convex subset  $C$  of a uniformly convex Banach space  $X$  into itself, then for any  $\alpha \in (0,1)$  the mapping  $U_\alpha$  is asymptotically regular.

##### 5. Demicompact nonexpansive mappings and successive approximations

Theorem 2.6 does not answer the question as to whether the sequence of successive approximations formed for the mapping  $U_\alpha$  is convergent or not. Under additional assumptions on  $U$  the positive answer to this question will follow from the theory of



demicompact mappings.

Definition 2.3 (Petryshyn [19]). A mapping  $U:C \rightarrow X$  of a subset  $C$  of a Banach space  $X$  into  $X$  is said to be demicompact if whenever  $\{x_n\} \subset C$  is a bounded sequence and  $\{x_n - Ux_n\}$  is a convergent sequence, then there exists a subsequence  $\{x_{n_i}\}$  which is convergent.

When  $C$  lies in a finite dimensional subspace of  $X$ , the condition of the demicompactness is automatically satisfied. Similarly, this condition is fulfilled whenever  $C$  is a compact subset of  $X$ .

The requirement of the demicompactness seems to be very restrictive. It turns out, however, that it is still weak enough in order to be satisfied for some broad classes of mappings, as it is shown by the following:

Proposition 2.2 (Petryshyn [19]). Each of the following conditions is sufficient for a mapping  $U:C \rightarrow X$  to be demicompact:

- (a)  $U$  is compact, i.e. maps bounded subsets of  $C$  into relatively compact subsets of  $X$ ;
- (b) the range  $R(I-U)$  is closed and the inverse mapping  $(I-U)^{-1}$  exists and is continuous;
- (c)  $X$  is a Hilbert space and for any  $x, y$  in  $C$ :  

$$\operatorname{Re} (Ux - Uy, x - y) \leq \frac{1}{2} \|x - y\|^2;$$
- (d)  $X$  is a Hilbert space and for any  $x, y$  in  $C$ :  

$$\operatorname{Re} (Ux - Uy, x - y) \leq \frac{1}{2} \|Ux - Uy\|^2.$$

Proof. In the case (a) the assertion is trivial since if the sequence  $\{x_n\}$  is bounded, then choosing from  $\{Ux_n\}$  a convergent subsequence  $\{U_{n_i}x_{n_i}\}$  one obtains a convergent sequence  $\{x_{n_i}\} = \{(x_{n_i} - Ux_{n_i}) + Ux_{n_i}\}$ .

In the case (b) the assertion is also trivial: if the sequence  $\{x_n - Ux_n\}$  is convergent, then so is the sequence  $\{x_n\} = \{(I - U)^{-1}(x_n - Ux_n)\}$ .

In the case (c) observe that if  $\{x_n - Ux_n\}$  is a Cauchy sequence, then by the inequality

$$\begin{aligned} \|(x_m - Ux_n) - (x_m - Ux_n)\|^2 &= \|(x_m - x_n) - (Ux_m - Ux_n)\|^2 \\ &= \|x_m - x_n\|^2 - 2\operatorname{Re}(x_m - x_n, Ux_m - Ux_n) + \|Ux_m - Ux_n\|^2 \geq \|Ux_m - Ux_n\|^2 \end{aligned}$$

the sequence  $\{Ux_n\}$  is also a Cauchy sequence. As in the case (a) this implies that the sequence  $\{x_n\}$  is convergent.

Finally, in the case (d) the assertion follows immediately from the inequality

$$\begin{aligned} \|(x_m - Ux_n) - (x_m - Ux_n)\|^2 &= \|(x_m - x_n) - (Ux_m - Ux_n)\|^2 \\ &= \|x_m - x_n\|^2 - 2\operatorname{Re}(x_m - x_n, Ux_m - Ux_n) + \|Ux_m - Ux_n\|^2 \geq \|x_m - x_n\|^2. \end{aligned}$$

Trivial examples in a one dimensional space show that there is no connection between the demicompactness and the continuity of mappings. Nevertheless, mappings which are simultaneously

demicompact and continuous have an important topological property expressed in the following:

Lemma 2.1. If a mapping  $U:C \rightarrow X$  is continuous and demicompact, then the mapping  $I-U$  maps closed bounded subsets of  $C$  into closed subsets of  $X$ .

Proof. Let  $D$  be a closed bounded subset of  $C$ . If  $y$  is in  $\overline{(I-U)(D)}$ , then there exists a sequence  $\{x_n\} \subset D$  such that  $x_n - Ux_n \rightarrow y$  as  $n \rightarrow \infty$ . By the demicompactness of  $U$ , we may assume that the sequence  $\{x_n\}$  converges to an element of  $D$ , say, to  $x$ . By the continuity of  $U$  it follows that  $x_n - Ux_n \rightarrow x - Ux$  so that  $y = x - Ux$ . This means that  $y$  is in  $(I-U)(D)$  and completes the proof.

Let us observe that the identical mapping  $I:X \rightarrow X$  of a Banach space  $X$  onto itself is demicompact if and only if  $X$  is of finite dimension. On the other hand, the mapping  $I-I$  trivially maps closed sets into closed sets. Therefore, the statement made in [6] that the demicompactness of  $U$  is equivalent to the requirement that  $I-U$  maps bounded closed sets into closed sets is incorrect, even for continuous mappings.

Combining the asymptotic regularity with the demicompactness or, more generally, with the consequence of the demicompactness and the continuity expressed in Lemma 2.1, we obtain the following general criterion for the convergence of the sequence of successive approximations.

Theorem 2.7 (Browder and Petryshyn [6]). Let  $C$  be a closed subset of a Banach space  $X$ . If the mapping  $U:C \rightarrow C$  is nonexpansive, asymptotically regular and the mapping  $I-U$  maps closed bounded subsets of  $C$  into bounded subsets of  $X$  (thus, in particular, if  $U$  is demicompact) and if the set  $F$  of fixed points of  $U$  in  $C$  is nonempty, then for any  $x$  in  $C$  the sequence  $\{U^n x\}$  is convergent to a fixed point of  $U$ .

If, in addition,  $C$  is bounded or, more generally, if there exists an  $x_0$  in  $C$  such that the sequence  $\{U^n x_0\}$  contains a bounded subsequence, then the assumption that  $F$  is a nonempty set follows from the remaining assumptions and, therefore, may be omitted.

Proof. For a  $y$  in  $F$  and any  $x$  in  $C$ , the sequence  $\{\|y - U^n x\|\}$  is nonincreasing. Hence, the sequence  $\{U^n x\}$  is bounded. Denote by  $D$  the closure of the set  $\{U^n x: n = 1, 2, \dots\}$ . From the asymptotic regularity of  $U$  it follows that  $(I-U)(U^n x) \rightarrow 0$  as  $n \rightarrow \infty$ , so that  $0$  belongs to the closure of the set  $(I-U)(D)$  and, therefore, to the set  $(I-U)(D)$  itself since  $D$  is closed. This means that there is a subsequence  $\{U^{n_i} x\}$  which converges to, say,  $y_0$  such that  $(I-U)y_0 = 0$ ; i.e.,  $y_0 = Uy_0$ . Since the sequence  $\{\|y_0 - U^n x\|\}$  does not increase,  $U^n x \rightarrow y_0$  as  $n \rightarrow \infty$ . To complete the proof, it suffices to observe that if for some  $x_0$  in  $C$  the sequence  $\{U^{n_i} x_0\}$  is bounded, then from the inequality

$$\|U^{n_i} x - U^{n_i} x_0\| \leq \|x - x_0\| \quad (i = 1, 2, \dots)$$

it follows that the sequence  $\{U^{n_i}x\}$  is also bounded, for any  $x$  in  $C$ , and then to apply the above argument to the set  $\{U^{n_i}x: i = 1, 2, \dots\}$  instead of the set  $\{U^n x: n = 1, 2, \dots\}$ .

The second part of Theorem 2.7 may be considered as equivalent to the statement that for a closed set  $C$  in a Banach space  $X$  a nonexpansive mapping  $U: C \rightarrow C$  which is asymptotically regular and such that  $I-U$  maps bounded closed subsets of  $C$  into closed subsets of  $X$  has a fixed point if and only if there exists in  $C$  an  $x_0$  such that the sequence  $\{U^n x_0\}$  contains a bounded subsequence. For arbitrary nonexpansive mappings but under additional assumptions on the set  $C$  and the space  $X$  a somewhat similar property is stated in the following:

Proposition 2.3 (Browder and Petryshyn [6]). Let  $C$  be a closed convex subset of a uniformly convex Banach space  $X$ . A nonexpansive mapping  $U: C \rightarrow C$  has a fixed point in  $C$  if and only if there exists an  $x_0$  in  $C$  such that the sequence  $\{U^n x_0\}$  is bounded (or, equivalently, if and only if the sequence  $\{U^n x\}$  is bounded for every  $x$  in  $C$ ).

Proof. The necessity is trivial. The sufficiency follows immediately from Corollary 2.1. Finally, the condition in the bracket is an obvious consequence of the nonexpansivity of  $U$ .

Coming back to Theorem 2.6 we are now able to prove the following criterion of the convergence of the modified sequence of successive approximations.

Theorem 2.8 (Browder and Petryshyn [6]). Let  $U: C \rightarrow C$  be a nonexpansive mapping of a closed convex set  $C$  in a uniformly convex Banach space  $X$  into itself. For  $\alpha \in (0,1)$ , let  $U_\alpha$  be defined by formula (2.4). If the mapping  $I-U$  maps closed bounded subsets of  $C$  into closed subsets of  $X$  and if the set  $F$  of fixed points of  $U$  is nonempty, then for any  $\alpha \in (0,1)$  and every  $x$  in  $C$  the sequence  $\{U_\alpha^n x\}$  is convergent to a fixed point of  $U$ .

Proof. By Theorem 2.6,  $U_\alpha$  is a nonexpansive asymptotically regular mapping of  $C$  into itself with the same set of fixed points as  $U$ . From the identity

$$(2.7) \quad I-U_\alpha = (1-\alpha)(I-U)$$

it follows that  $I-U_\alpha$  maps bounded closed subsets of  $C$  into closed subsets of  $X$ . Thus, a straightforward application of Theorem 2.7 completes the proof.

It should be noticed that if  $C$  is a bounded set, then the assumption that the set  $F$  is nonempty follows directly from Theorem 2.2 and, therefore, may be omitted.

From Lemma 2.1 it follows that Theorem 2.8 can be applied, in particular, to demicompact nonexpansive mappings. Hence, by Proposition 2.3, it can be applied as well to compact nonexpansive mappings. For the latter we obtain in this way the following:

Corollary 2.3 (Krasnosel'skii [16]). Let  $U:C \rightarrow C$  be a nonexpansive mapping of a closed convex set  $C$  in a uniformly convex Banach space  $X$  into itself. If the set  $U(C)$  is relatively compact, then for any  $\alpha \in (0,1)$  and every  $x$  in  $C$  the sequence  $\{U_\alpha^n x\}$  converges to a fixed point of  $U$ .

Proof. The convex closure  $D$  of the set  $U(C)$  is a compact subset of  $C$ , obviously invariant under  $U$ . The restriction of  $U$  to  $D$  satisfies the assumptions of the Schauder fixed point theorem (and also the assumptions of Theorem 2.2) so that  $U$  has at least one fixed point in  $D$ . Since  $U$  is a compact mapping, an application of Theorem 2.8 completes the proof.

A nontrivial generalization of Corollary 2.3 is given by the following:

Theorem 2.9 (Edelstein [11]). If  $U:C \rightarrow C$  is a nonexpansive mapping of a compact set  $C$  in a strictly convex Banach space  $X$  into itself, then for any  $\alpha \in (0,1)$  and every  $x$  in  $C$  the sequence  $\{U_\alpha^n x\}$  is convergent to a fixed point of  $U$ .

Proof. By the Schauder fixed point theorem, the set  $F$  of fixed points of  $U$  is nonempty. By Proposition 2.1,  $F$  is a convex closed and hence compact subset of  $C$ .

Observe first that for  $x$  in  $C$  and  $y$  in  $F$  the relationship

$$\|x-y\| = \|U_\alpha x-y\|$$

is possible if and only if  $x$  is a fixed point. Indeed, from

$$\|x-y\| = \|U_{\alpha}x-y\| = \|\alpha(x-y)+(1-\alpha)(Ux-y)\| \leq \alpha\|x-y\|+(1-\alpha)\|Ux-y\| \leq \|x-y\|$$

it follows that

$$\|\alpha(x-y)+(1-\alpha)(Ux-y)\| = \alpha\|x-y\|+(1-\alpha)\|Ux-y\|$$

and hence, by the strict convexity of  $X$ ,  $Ux = x$  so that

$$U_{\alpha}x = x.$$

The functions

$$\Phi(x) = \min \{\|x-y\| - \|U_{\alpha}x-y\| : y \in F\}, \quad \Psi(x) = \min \{\|x-y\| : y \in F\}$$

are continuous on  $C$ , nonnegative and equal zero if and only if  $x \in F$ . Now, for a given  $x$  in  $C$ , the sequence  $\{\Psi(U_{\alpha}^n x)\}$  is non-increasing. If its limit were positive, then by the compactness of  $C$  the sequence  $\{\Phi(U_{\alpha}^n x)\}$  would be bounded from below by a positive constant, i.e. we would have

$$\|U_{\alpha}^{n+1}x-y\| \leq \|U_{\alpha}^n x-y\| - \delta$$

for some  $\delta > 0$  and  $n = 0, 1, \dots$  which is clearly impossible.

Thus,  $\Psi(U_{\alpha}^n x) \rightarrow 0$  as  $n \rightarrow \infty$  and this means that the sequence  $\{U_{\alpha}^n x\}$  is necessarily convergent to an element of  $F$  and completes the proof.



## 6. Weak convergence of successive approximations

So far we were concerned with the strong convergence of the sequence of successive approximations. In what follows we shall discuss similar problems for the weak convergence, mainly under the additional assumption that  $X$  is a Hilbert space.

Definition 2.3 (Browder and Petryshyn [6]). Let  $C$  be a subset of a Banach space  $X$ . A mapping  $U:C \rightarrow X$  is called demiclosed if for any sequence  $\{x_n\} \subset C$  which converges weakly to an  $x$  in  $C$  the strong convergence of the sequence  $\{Ux_n\}$  to a  $y$  in  $X$  implies that  $Ux = y$ .

In other words, the mapping  $U:C \rightarrow X$  is demiclosed if its graph in  $C \times X$  is closed in the Cartesian product topology induced in  $C \times X$  by the weak topology in  $C$  and the strong topology in  $X$ .

From this definition it follows, in particular, that a mapping  $U:C \rightarrow X$  which is weakly continuous, i.e. is continuous from the weak topology of  $C$  to the weak topology of  $X$ , is necessarily demiclosed.

Similarly, the following statement is an immediate consequence of the above definition.

Proposition 2.4 (Browder and Petryshyn [6]). Let  $C$  be a closed convex set in a Banach space  $X$ . Suppose that the mapping  $U:C \rightarrow C$  is asymptotically regular and that the mapping  $I-U$  is demiclosed. Then for every  $x$  in  $C$  the weak limit of any weakly

convergent subsequence of the sequence  $\{U^n x\}$  is a fixed point of  $U$ .

In particular, if  $X$  is reflexive and  $U$  has exactly one fixed point  $y$ , then for every  $x$  in  $C$  the sequence  $\{U^n x\}$  converges weakly to  $y$ .

Proof. By Theorem 1.1, the weak limit  $x_0$  of a weakly convergent sequence  $\{U^{n_i} x\}$  lies in  $C$ . By the asymptotic regularity, the sequence  $\{(I-U)(U^{n_i} x)\}$  is convergent to zero as  $i \rightarrow \infty$  so that, by the demiclosedness of  $I-U$ , we have  $(I-U)x_0 = 0$ , i.e.  $Ux_0 = x_0$ .

If  $U$  has at least one fixed point, then for any  $x$  in  $C$  the sequence  $\{U^n x\}$  is bounded. Therefore, when  $X$  is reflexive, there exists a subsequence of the sequence  $\{U^n x\}$  which is weakly convergent. If, in addition,  $U$  has exactly one fixed point  $y$ , then every weakly convergent subsequence of  $\{U^n x\}$  converges weakly to  $y$  and this means that the sequence  $\{U^n x\}$  itself converges weakly to  $y$ .

Lemma 1.3 enables us to state the following useful property of nonexpansive mappings in Hilbert spaces.

Proposition 2.5 (Browder [3]). In a Hilbert space  $X$  for every nonexpansive mapping  $U:C \rightarrow X$  ( $C \subset X$ ) the mapping  $I-U$  is demiclosed.

Proof. Let  $\{x_n\} \subset C$  be a sequence which is weakly convergent to an element  $x_0$  of  $C$  and let  $x_n - Ux_n \rightarrow y_0$  as  $n \rightarrow \infty$ .

Then we have

$$\liminf_{n \rightarrow \infty} \|x_n - x_0\| \geq \liminf_{n \rightarrow \infty} \|Ux_n - Ux_0\| = \liminf_{n \rightarrow \infty} \|x_n - y_0 - Ux_0\|.$$

Hence, by Lemma 1.3,  $x_0 - Ux_0 = y_0$ .

Using Proposition 2.5 we shall prove now the following:

Theorem 2.10 (Opial [18]). Let  $C$  be a closed convex set in a Hilbert space  $X$  and  $U: C \rightarrow C$  a nonexpansive asymptotically regular mapping for which the set  $F$  of fixed points is nonempty. Then for every  $x$  in  $C$  the sequence  $\{U^n x\}$  is weakly convergent to a fixed point of  $U$ .

Proof. For every  $y$  in  $F$ , there exists the nonnegative limit

$$d(y) = \lim_{n \rightarrow \infty} \|U^n x - y\|.$$

Furthermore, for any  $d \geq 0$  the set

$$F_d = \{y \in F : d(y) \leq d\}$$

is a convex closed and bounded subset of  $F$ , nonempty if  $d$  is large enough. Therefore, since  $X$  is reflexive, there exists the smallest  $\delta \geq 0$  for which the set  $F_\delta$  is nonempty.  $F_\delta$  consists of exactly one element, say  $y_0$ , since otherwise the midpoint of

the segment joining two distinct elements of  $F$  (which belongs to  $F$ , by Proposition 2.1) would belong, by the uniform convexity of  $X$ , to an  $F_d$  with  $d < \delta$ .

We shall prove that the sequence  $\{U^n x\}$  converges weakly to  $y_0$ . Suppose the contrary. Then there exists a weakly convergent subsequence  $\{U^{n_i} x\}$  whose weak limit, say  $y_1$ , is different from  $y_0$ . By Propositions 2.4 and 2.5  $y_1$  is an element of  $F$ . On the other hand, by Lemma 1.3, we have

$$\delta = d(y_0) = \lim_{i \rightarrow \infty} \|U^{n_i} x - y_0\| > \lim_{i \rightarrow \infty} \|U^{n_i} x - y_1\| = d(y_1)$$

which yields a contradiction with the definition of  $\delta$  and completes the proof.

By Theorem 2.6, Theorem 2.10 applies in particular to the modified sequence of successive approximations so that we have the following result which for weakly continuous nonexpansive mappings has been proved in [21]:

Theorem 2.11. Let  $C$  be a closed convex subset of a Hilbert space  $X$  and  $U: C \rightarrow C$  a nonexpansive mapping with a non-empty set of fixed points. For an  $\alpha \in (0,1)$ , let  $U_\alpha$  be defined by (2.4). Then for every  $x$  in  $C$  the sequence  $\{U_\alpha^n x\}$  is weakly convergent to a fixed point of  $U$ .

It should be noticed that if  $C$  is bounded, then by Theorem 2.2 the existence of a fixed point of  $U$  follows from the nonexpansivity of  $U$  so that then in Theorem 2.11 the assumption

that the set  $F$  is nonempty may be omitted.

In conclusion, let us observe that all results of §§4-6 can be applied to equations of the type

$$(2.8) \quad x - Ux = y$$

where  $U: X \rightarrow X$  is a nonexpansive mapping of a Banach space  $X$  into itself and  $y$  is a given element of  $X$ . To this end, it is sufficient to consider in place of  $U$  the mapping  $U_y: X \rightarrow X$  defined by  $U_y x = y + Ux$ , since every fixed point of  $U_y$  is a solution equation (2.8) and conversely.

#### 7. Contractive approximations of nonexpansive mappings

Let  $C$  be a convex closed subset of a Banach space  $X$  and  $U: C \rightarrow C$  a nonexpansive mapping. For any  $k \in [0, 1)$  and any  $x_0$  in  $C$ , the mapping

$$U_k x = kUx + (1-k)x_0$$

maps  $C$  into itself and is contractive with the Lipschitz constant equal to  $k$ . For  $k$  sufficiently close to 1,  $U_k$  is thus a contractive approximation of the mapping  $U$ .

By the Banach contraction principle, for any  $k \in [0, 1)$  there exists in  $C$  a unique fixed point  $x_k$  of the mapping  $U_k$ ;

$$x_k = kUx_k + (1-k)x_0.$$

Is  $x_k$ , for  $k$  sufficiently close to 1, a good approximation of a fixed point of  $U$ ? A partial affirmative answer to this question is given by the following:

Theorem 2.12 (Browder [5]). Suppose that  $X$  is a Hilbert space and that the set  $F$  of fixed points of the nonexpansive mapping  $U: C \rightarrow C$  is nonempty. Then

$$\lim_{k \rightarrow 1} x_k = y_0,$$

where  $y_0$  is the fixed point of  $U$  closest to  $x_0$ .

Proof. First of all let us observe that, by Proposition 2.1,  $F$  is a closed convex subset of  $C$  so that the point  $y_0$  exists and is uniquely determined.

Without loss of generality we may assume that  $x_0 = 0$ . We have then

$$\|x_k/k - y_0\|^2 = \|Ux_k - Uy_0\|^2 \leq \|x_k - y_0\|^2.$$

Hence

$$\|x_k\|^2 - 2k \operatorname{Re} (x_k, y_0) + k^2 \|y_0\|^2 \leq k^2 (\|x_k\|^2 - 2 \operatorname{Re} (x_k, y_0) + \|y_0\|^2)$$

and finally, after simple cancellations,

$$(1+k) \|x_k\|^2 \leq 2k \operatorname{Re} (x_k, y_0).$$

Since  $k < 1$ , we have therefore  $\|x_k\|^2 \leq \operatorname{Re} (x_k, y_0)$  and hence

$$(2.9) \quad \|x_k\| \leq \|y_0\| \quad (0 \leq k < 1).$$

Suppose now that  $\{x_i\} = \{x_{k_i}\}$  with  $k_i \rightarrow +\infty$  as  $i \rightarrow +\infty$  is a weakly convergent sequence and let  $x$  be its limit. Since

$$\lim_{i \rightarrow \infty} \|x_i - Ux_i\| = \lim_{i \rightarrow \infty} (1 - k_i)k_i \|x_i\| = 0$$

( $k_i \rightarrow 1$  and the sequence  $\{\|x_i\|\}$  is bounded), from Proposition 2.5 it follows that  $x$  is a fixed point of  $U$ . From inequality (2.9) we have, in the limit,  $\|x\| \leq \|y_0\|$  so that  $\|x\| = \|y_0\|$  and hence  $x = y_0$ . Since in the relationship

$$\|y_0\|^2 \geq \|x_i\|^2 = \|x_i - y_0\|^2 + \|y_0\|^2 + 2\operatorname{Re}(x_i - y_0, y_0)$$

the last term goes to zero as  $i \rightarrow +\infty$ , we conclude that the sequence  $\{x_i\}$  is strongly convergent to  $y_0$ .

To complete the proof, suppose now that  $x_k$  does not converge strongly to  $y_0$  as  $k \rightarrow \infty$ . Then there exists a sequence  $\{x_{k_i}\}$  with  $k_i \rightarrow \infty$  such that none of its subsequences is convergent to  $y_0$ . But by the reflexivity of the space  $X$  we can always choose from  $\{x_{k_i}\}$  a weakly convergent subsequence and such a subsequence, as we have shown, is necessarily strongly convergent to  $y_0$ . This contradiction completes the proof.

## 8. Extensions of nonexpansive mappings

Can a nonexpansive mapping  $U: C \rightarrow X$  of a subset  $C$  of a Banach space  $X$  into  $X$  be extended to a nonexpansive mapping of  $X$  into itself? Since the pioneering work of Kirszbraun [14] this natural question (in much more general setting - for nonexpansive mappings of subsets of a metric space  $X$  into another metric space

$X'$ ) has been extensively discussed by several authors (for an extensive bibliography of this subject see a recent expository paper [7]). Here we shall confine ourselves to showing that the answer to this question is positive if  $X$  has the structure of a Hilbert space.

The key role in our discussion will be played by the following:

Theorem 2.13 (Kirszbraun [14]). If  $x_1, \dots, x_m, x'_1, \dots, x'_m, p$  are points of a finite dimensional Euclidean space  $X$  such that

$$(2.10) \quad \|x'_i - x_j\| \leq \|x_i - x_j\| \quad (i, j = 1, \dots, m),$$

then there exists in  $X$  a point  $p'$  such that

$$\|x'_i - p'\| \leq \|x_i - p\| \quad (i = 1, \dots, m).$$

In a more geometrical manner, this theorem may be stated as follows. Let  $B_i = B(x_i, r_i)$ ,  $B'_i = B(x'_i, r_i)$  ( $i = 1, \dots, m$ ) be  $2m$  balls in a finite dimensional Euclidean space  $X$ . Then if for the distances of their inequalities (2.10) hold and if the intersection of  $B_i$  ( $i = 1, \dots, m$ ) is nonempty, then so is the intersection of  $B'_i$  ( $i = 1, \dots, m$ ).

Proof (Schoenberg [22]). For every  $\lambda \geq 0$ , the set

$$P_\lambda = \{p' \in X : \|x'_i - p'\| \leq \lambda \|x_i - p\| \quad (i = 1, \dots, m)\}$$

is bounded, closed and nonempty if  $\lambda$  is sufficiently large. More-



over,  $\mu \leq \lambda$  implies that  $P_\mu \subset P_\lambda$ . Therefore, there exists the smallest nonnegative number  $\alpha$  for which the set  $P_\alpha$  is nonempty. If  $\alpha \leq 1$ , then the assertion of Theorem 2.13 is obviously true.

Suppose that  $\alpha > 1$  and let  $p'$  be an element of  $P_\alpha$ . Without loss of generality we can assume that

$$(2.11) \quad \begin{aligned} \|x'_i - p'\| &> \|x_i - p\| & (i = 1, \dots, k), \\ \|x'_i - p'\| &\leq \|x_i - p\| & (i = k+1, \dots, m). \end{aligned}$$

The element  $p'$  lies in the convex hull of the set  $\{x'_1, \dots, x'_k\}$  since otherwise it would be possible to move  $p'$  slightly (in the direction perpendicular to any hyperplane separating  $\{x'_1, \dots, x'_k\}$  and  $p'$ ) in a manner to decrease all distances  $\|x'_i - p'\|$  ( $i = 1, \dots, k$ ) and thus to decrease  $\alpha$  itself. Therefore,

$$(2.12) \quad p' = \sum_{i=1}^k \lambda_i x'_i \quad (\lambda_1, \dots, \lambda_k \geq 0; \sum_{i=1}^k \lambda_i = 1).$$

For  $i, j = 1, \dots, k$  we have

$$\|x_i - x_j\|^2 = \|x_i - p + p - x_j\|^2 = \|x_i - p\|^2 + \|x_j - p\|^2 - 2(x_i - p, x_j - p)$$

and similarly

$$\|x'_i - x'_j\|^2 = \|x'_i - p'\|^2 + \|x'_j - p'\|^2 - 2(x'_i - p', x'_j - p').$$

Combining these relations with (2.10) and (2.11), we obtain

$$(x'_i - p', x'_j - p') > (x_i - p, x_j - p) \quad (i, j = 1, \dots, k).$$

Therefore, by (2.12), we have

$$\begin{aligned} 0 &= \left\| \sum_{i=1}^k \lambda_i x'_i - p' \right\|^2 = \left\| \sum_{i=1}^k \lambda_i (x'_i - p') \right\|^2 = \sum_{i,j=1}^k \lambda_i \lambda_j (x'_i - p', x'_j - p') \\ &> \sum_{i,j=1}^k \lambda_i \lambda_j (x_i - p, x_j - p) = \left\| \sum_{i=1}^n \lambda_i (x_i - p) \right\|^2. \end{aligned}$$

This obvious contradiction completes the proof.

Now, using the standard compactness argument, we are able to extend this result to arbitrary Hilbert spaces and infinite families of balls.

Theorem 2.14 (Valentine [23]). Let  $\{B_\alpha\}_{\alpha \in A}$ ,  $\{B'_\alpha\}_{\alpha \in A}$ ,  $B_\alpha = B(x_\alpha, r_\alpha)$ ,  $B'_\alpha = B(x'_\alpha, r'_\alpha)$  be two families of balls in a (real or complex) Hilbert space  $X$ . If

$$\|x'_\alpha - x'_\beta\| \leq \|x_\alpha - x_\beta\| \quad (\alpha, \beta \in A)$$

and the intersection  $\bigcap_{\alpha \in A} B_\alpha$  is nonempty, then so is the intersection  $\bigcap_{\alpha \in A} B'_\alpha$ .

Proof. Choose an index  $\alpha_0$  in  $A$ . By Theorem 1.4, the ball  $B'_0 = B'_{\alpha_0}$  is weakly compact. For every  $\alpha \in A$ , the intersection  $B'_\alpha \cap B'_0$  is a closed and convex subset of  $B'_0$  and, therefore, by

Theorem 1.1, it is a weakly closed subset of  $B'_0$ . Hence, to complete the proof, it suffices to show that for every finite system of indexes  $\alpha_1, \dots, \alpha_m$  in  $A$  the balls  $B'_i = B'_{\alpha_i}$  ( $i = 1, \dots, m$ ) and  $B'_0$  have a nonempty intersection.

To prove this, in turn, it suffices to consider the finite dimensional subspace  $X'$  of  $X$  spanned by the centers of the balls  $B'_i$ ,  $B_i = B_{\alpha_i}$  ( $i = 0, \dots, m$ ). If  $X$  is a real Hilbert space, then  $X'$  is a finite dimensional Euclidean space. If  $X$  is a complex Hilbert space, then the finite dimensional complex Hilbert space  $X'$  with the scalar product  $(,)$  is isometric to  $X'$  endowed with the scalar product  $\langle, \rangle = \operatorname{Re} (,)$ . So, in both cases we can consider  $X'$  as a finite dimensional real Euclidean space.

By assumptions, the balls  $X' \cap B_i$  ( $i = 0, \dots, m$ ) have a nonempty intersection. By Theorem 2.13, the balls  $X' \cap B'_i$  ( $i = 0, \dots, m$ ) have also a nonempty intersection. Therefore, so do the balls  $B'_i$  ( $i = 0, \dots, m$ ) and the proof is completed.

Applying Theorem 2.14, we can now easily state the following general theorem on the existence of nonexpansive extensions for nonexpansive mappings, proved by Kirszbraun [14] for finite dimensional Euclidean spaces and then extended by Valentine [23] to arbitrary Hilbert spaces.

Theorem 2.15. Let  $u: C \rightarrow X$  be a nonexpansive mapping of a subset  $C$  of a Hilbert space  $X$  into  $X$ . There exists a nonexpansive mapping  $U: X \rightarrow X$  such that its restriction to  $C$  is identical with  $u$ .

Proof. Suppose that  $U$  is already defined in a set  $C \subset C'$ ,  $C' \neq X$ . Let  $p \in X \setminus C'$ . Applying the preceding theorem to the families of balls

$$\{B(x, \|x-p\|) : x \in C'\} \quad \{B(Ux, \|x-p\|) : x \in C'\}$$

we can choose a point  $p'$  in  $X$  such that

$$\|Ux - p'\| \leq \|x - p\|$$

for every  $x$  in  $C'$ . Setting  $Up = p'$ , we obtain a nonexpansive mapping of the set  $C' \cup \{p\}$  into  $X$ .

It is now clear that the usual procedure based on the Kuratowski-Zorn lemma will complete the proof.

Let us observe that applying Theorem 2.15 to the mapping  $u = \frac{1}{L}v$  we can conclude that for every subset  $C$  of a Hilbert space  $X$  and every mapping  $v: C \rightarrow X$  satisfying the Lipschitz condition

$$\|v(x) - v(y)\| \leq L\|x - y\| \quad (x, y \in C)$$

there exists a mapping  $V: X \rightarrow X$  satisfying the Lipschitz condition with the same constant  $L$  and identical with  $v$  on  $C$ .

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### Chapter III

## MONOTONE MAPPINGS IN BANACH SPACES

#### 1. Introduction

The theory of monotone mappings in Banach spaces is of a very recent origin. Some special results which now can be stated or interpreted in terms of this theory were obtained in the early 1950s for gradient mappings considered in the calculus of variations in Banach spaces and were presented in the book of Vainberg [52] in the context of the theory of variational methods for the study of nonlinear operators and equations. In the late 1950s, still in the spirit of Vainberg's book, further new "fixed-point principles" were established by Krasnosel'skiĭ [34], Vainberg and Kačurovskiĭ [54] and Kačurovskiĭ [27], [28].

The explicit definition of the monotone mapping of a Banach space into its dual space which arose in a natural way from these investigations and was first introduced in a short note of Kačurovskiĭ [29] (see also [30]) along with some simple properties of such mappings would have been only a formal and sterile step toward an apparently more general but shallow and practically useless theory if it were not followed (in the logical sense; actually preceded by a few months) by the announcement in a short note of Vainberg [53] of the first fixed point theorem for monotone mappings in Hilbert spaces satisfying a Lipschitz condition. This clearly showed that the theory of monotone mappings need not be restricted to gradient mappings and can be based on more primary structural properties of normed spaces.

About the same time but independently, monotone mappings in



Hilbert spaces were studied by Zarantonello [55] who proved that if  $T$  is a monotone mapping of a Hilbert space  $H$  into itself and satisfied a Lipschitz condition, then the mapping  $I+T$  maps  $H$  onto  $H$  - a result which only formally differs from that of Vainberg.

The turning point in the development of the theory of monotone mappings was the extension of the Vainberg-Zarantonello theorem to continuous monotone mappings in 1962 by Minty [42] whose success was largely due to a skillful treatment of the problem in the context of a still more primitive mathematical structure, that of a simple monotonicity relation induced in the space  $H \times H$  by the scalar product in the Hilbert space  $H$ . In his study, the natural relationship between monotone mappings of  $H$  into itself and monotone subsets of the space  $H^2$  along with the assumption of continuity proved quite adequate to yield sound conceptual foundations of the new abstract theory. In addition, exhibiting an intimate relation between nonexpansive and monotone mappings, which in the works of Vainberg and Zarantonello appeared as a connection between contractive and monotone Lipschitzian mappings, Minty awoke the interest in nonexpansive mappings and, in particular, revived the Kirschbraun-Valentine theorem which at that time, lacking serious applications, seemed to be doomed to oblivion.

One year later the Minty theorem was proved by Browder [1] under a weaker condition on monotonicity and further, in a series of three notes [2]-[4], the latter weakened step by step the continuity requirements up to a strangely weak assumption of the continuity from

line segments in  $H$  to the weak topology in  $H$ .

The next important step in the development of the theory of monotone mappings was its extension to reflexive Banach spaces carried out by Browder [5] for separable reflexive spaces and, slightly later but independently, by Minty [43] without a separability assumption. Together with the above mentioned notes of Browder, this generalization laid down the topological foundations of the theory, became a typical model for further extensions and, last but not least, allowed one to embody in its general setting special situations encountered in the abstract calculus of variations.

Further generalizations to densely defined mappings and multi-valued mappings were given by Browder in [6], [8] (see also [9], [16]) and [18], respectively. Recently, an analogous extension to a class of nonreflexive Banach spaces was given by Browder [20].

In 1964, the joint effort of Kato [32] and Browder [10] shed a bright light on the connections between monotonicity of a mapping and various continuity assumptions. From the conceptual viewpoint the most important of their results seems to be a theorem of Kato stating that in finite dimensional spaces the monotonicity of a mapping and its continuity from line segments imply the continuity. It explains that, when combined with the monotonicity, this last continuity assumption is not so extremely weak as it might seem.

Some of the basic results of the theory of monotone mappings

in Banach spaces use very little of the normed space structure and have, as pointed out in several places by Browder [7], [14], [15] (see also [25]), natural generalizations to locally convex linear spaces.

A new direction in the development of the theory of monotone mappings - the study of nonlinear variational inequalities for mappings defined on convex closed subsets of a Banach space, was originated by a recent note of Browder [12] and a paper of Hartman and Stampacchia [26] as a nonlinear generalization of linear variational inequalities studied by Stampacchia [51], Lions and Stampacchia [39] and Lescarret [37].

In his notes [22] and [23] Browder made an attempt to elaborate a unified approach to both the theory of monotone mappings and that of nonlinear variational inequalities, as well as to the theory of direct methods of the calculus of variations in Banach spaces.

The variational methods of the theory of nonlinear operators have been finding for years their most important applications in the theory of nonlinear integral and partial differential equations. Their applicability, however, has been naturally restricted to problems with direct variational interpretations. In a natural way, the theory of monotone mappings - an abstract generalization of basic ideas of the variational methods - widened the class of such problems. And every successive extension of this theory broadened still further the domain of its applicability. Actually, from the very beginning, various attempts of the extension of the domain of

applicability have been a driving factor in the development of the abstract theory. So, for instance, its extension to densely defined mappings arose from the study of parabolic boundary value problems and the extension to nonreflexive spaces was obtained in an attempt to embody in the theory of monotone mappings some elliptic boundary value problems which could not be treated in the framework of reflexive spaces.

The possibility of application of the new abstract theory to partial differential equations was first grasped by Browder who in [1]-[5] and [7] attacked by these methods elliptic boundary value problems. Later on, his study of elliptic equations was continued by himself [11], [15], [20], Leray and Lions [36] and by Hartman and Stampacchia [26]. Applications to parabolic boundary value problems were given by Browder in [6] and [8]. Ideas of the theory of monotone mappings were also applied to hyperbolic systems and wave equations by Lions [38] and Lions and Strauss [40], [41]. Applications to nonlinear equations of evolution were given by Browder [9], [16] and Kato [33].

The extension of the theory to multi-valued mappings was applied by Browder [18] to the study of duality mappings in reflexive Banach spaces which in the framework of the theory of single-valued monotone mappings had been restricted in an earlier paper of Browder [17] to strictly convex reflexive spaces.

A comprehensive survey lecture on these various applications was delivered by Browder at the 17th Symposium of the American Mathematical Society in Applied Mathematics (New York,

April, 1964) and then published in [19].

Applications of the theory of nonlinear monotone mappings to integral equations were given by Zarantonello [55], Minty [44] and Dolph and Minty [24].

The main result of the theory of monotone mappings is contained in the statement that under very weak continuity assumptions and some additional conditions on the behavior at infinity each monotone mapping  $T$  of a reflexive Banach space  $X$  into its dual space  $X^*$  is necessarily surjective, i.e. maps  $X$  onto  $X^*$ . This means that for each given  $u_0$  in  $X^*$ , the functional equation  $Tx = u_0$  has a solution in  $X$ . One proves this basic property first in finite dimensional Euclidean spaces, usually by a simple index argument, and then carries it over to arbitrary reflexive Banach spaces by a weak compactness argument. Additional technical difficulties appear when mappings are defined only on subsets (convex or dense) of the space  $X$  or when more sophisticated monotonicity conditions are considered.

In this technical aspect, the theory of monotone mappings resembles that of compact mappings. This formal resemblance, however, goes much further as it is possible to extend in various forms (see Browder [13] and [14]) to monotone mappings the Borsuk antipodal theorem, the Leray-Schauder theorem on continuous continuation of fixed points and the Schauder theorem on invariance of domain. Even more, it turns out that some basic results of both theories can be treated in a unified form (see Browder [21]).

As in many other instances of similar type in functional

analysis, the transition from finite dimensional subspaces of a space  $X$  to the whole space carried through in proofs of main results of the theory of monotone mappings is non-constructive in nature. It becomes constructive, however, under additional separability and regularity conditions on  $X$  and acquires in this case many features of the orthogonal projection methods for solving linear functional equations in Banach spaces. This constructive aspect of the theory of monotone mappings was recently developed by Petryshyn [47]-[50] (see also Kaniel [31]) in the framework of the more general theory of so-called projectionally compact mappings.

## 2. Monotone sets

Let  $X$  be a Banach space over the field  $C$  of complex numbers and  $X^*$  its dual space, i.e. the space of all linear conjugate continuous mappings of  $X$  into  $C$ . In an obvious manner  $X$  may be considered as a Banach space  $\tilde{X}$  over the field  $R$  of real numbers.

For any  $u$  in  $X^*$ , the formula

$$(3.1) \quad (\tilde{u}, x) = \operatorname{Re} (u, x) \quad \text{for all } x \text{ in } X$$

defines a real linear functional  $\tilde{u}$  on  $\tilde{X}$ , i.e. an element of the dual space  $\tilde{X}^*$ . This correspondence  $J: u \rightarrow \tilde{u}$  is a one-to-one mapping of  $X^*$  onto  $\tilde{X}^*$  since for each given  $\tilde{u}$  in  $\tilde{X}^*$ , the formula

$$(3.2) \quad (u, x) = (\tilde{u}, x) + i(\tilde{u}, ix) \quad \text{for all } x \text{ in } X$$

defines an element  $u$  of  $X$  which satisfies (3.1). The mappings  $J$  and  $J^{-1}$  are both linear and, by (3.1) and (3.2),  $\|u\|/2 \leq \|Ju\| \leq \|u\|$  for all  $u$  in  $X^*$ .

This correspondence enables us to confine our study of monotone mappings to real Banach spaces. Analogous results for complex Banach spaces follow immediately from that study by a mere replacement of the form  $(u, x)$  by  $\operatorname{Re} (u, x)$  in all definitions and statements of this chapter.

Let  $X$  be a Banach space (over the field of real numbers) and  $X^*$  its dual space. For an  $x$  in  $X$  and a  $u$  in  $X^*$ ,  $[x, u]$  will stand for the correspondent element of the Cartesian product  $X \times X^*$ .

Definition 3.1. The set  $M \subset X \times X^*$  is called monotone if for any pair  $[x, u], [y, v]$  of elements of  $M$ ,

$$(3.3) \quad (u-v, x-y) \geq 0.$$

$M$  is said to be maximal monotone if it is monotone and maximal in the family of all monotone sets ordered by inclusion; i.e., if for any monotone set  $N$ ,  $M \subset N$  implies that  $M = N$ .

It is clear that, by the Kuratowski-Zorn lemma, for any monotone set  $M$  there exists a maximal monotone set which contains  $M$ . It is also clear that if  $M$  is a monotone (maximal monotone) set, then for any positive  $\lambda$  the set  $\lambda M = \{[x, \lambda u]: [x, u] \in M\}$  is also monotone (maximal monotone).

Proposition 3.1 (Minty [46], Browder [14]). If  $M$  is a maximal monotone set in  $X \times X^*$ , then for each  $u$  in  $X$ , the set

$$M_u = \{x \in X : [x, u] \in M\}$$

is a closed convex subset of  $X$ .

Proof. For each  $[y, v]$  in  $M$ , the set

$$M_u(y, v) = \{x \in X : (v - u, x - y) \geq 0\}$$

is closed and convex. Since  $M$  is maximal monotone,

$$M_u = \bigcap_{[y, v] \in M} M_u(y, v),$$

so that  $M_u$  is also closed and convex.

Let now  $X$  be a Hilbert space. Then  $X = X^*$  and we can define the mapping  $p: X^2 \rightarrow X$  setting  $p([x, u]) = u + x$  for all  $[x, u]$  in  $X^2$ .

Lemma 3.1. Let  $X$  be a Hilbert space. If  $M$  is a monotone set in  $X^2$ , then the mapping  $p: M \rightarrow X$  is one-to-one.

Proof. If  $[x, u], [y, v]$  are in  $M$  and  $u + x = y + v$ , then  $x - y = v - u$  so that

$$(u - v, x - y) = -\|x - y\|^2 = -\|u - v\|^2.$$



Hence from (3.3) it follows immediately that  $x = y$  and  $u = v$ .

Proposition 3.2 (Minty [42]). Let  $X$  be a Hilbert space. Then a monotone subset  $M$  of  $X^2$  is maximal monotone if and only if  $p(M) = X$ ; i.e., if and only if  $p$  maps  $M$  onto  $X$ .

Proof. If  $p(M) = X$ , then the maximality of  $M$  follows immediately from the preceding lemma.

To prove that the condition  $p(M) = X$  is also necessary for the maximality of  $M$ , we define an auxiliary mapping  $q: p(M) \rightarrow X$  as follows: by Lemma 3.1, for any  $z$  in  $p(M)$  there is a uniquely determined element  $[x, u]$  in  $M$  such that  $z = u + x$ ; we set  $q(z) = u - x$ . The mapping  $q$  is nonexpansive since from the identity

$$(3.4) \quad \|(u-x) - (v-y)\|^2 = \|(u+x) - (v+y)\|^2 - 4(u-v, x-y),$$

for  $[x, u], [y, v]$  in  $M$ , it follows that

$$(3.5) \quad \|(u-x) - (v-y)\| \leq \|(u+x) - (v+y)\|.$$

If  $p(M) \neq X$ , then by Theorem 2.15 we can extend  $q$  to a non-expansive mapping  $Q: X \rightarrow X$ . Taking  $z$  in  $X \setminus p(M)$  and setting

$$y = \frac{1}{2}(Q(z)+z), \quad v = \frac{1}{2}(Q(z)-z),$$

we easily verify that the set  $M \cup \{[y, v]\}$  is monotone: for any  $[x, u]$  in  $M$ , inequality (3.5) and relationship (3.4) imply (3.3)

This means that  $M$  cannot be maximal monotone if  $p(M) \neq X$  and completes the proof.

Let us remark that Lemma 3.1 is equivalent to the statement that if  $M$  is a monotone subset of  $X^2$ , then for any given  $z$  in  $X$ , the representation

$$z = u + x \quad \text{with } [x, u] \text{ in } M,$$

if it exists, is necessarily unique. Proposition 3.2 states that  $M$  is maximal monotone if and only if such a representation does exist for each  $z$  in  $X$ .

Lemma 3.1 and Proposition 3.2 have a very simple geometrical interpretation. To simplify the discussion, let us consider in some details the case  $X = \mathbb{R}$ . We can identify  $\mathbb{R}^2$  with the Euclidean plane. By Definition 3.1, a subset  $M$  of  $\mathbb{R}^2$  is monotone if and only if for any pair of points  $[x, u], [y, v]$  in  $M$ , the inequality  $x \leq y$  implies that  $u \leq v$ . For instance, the sets

$$M_1 = \{[n+x, n]: n = 0, \pm 1, \dots; 0 \leq x \leq 1\},$$

$$M_2 = \{[n, n+x-1]: n = 0, \pm 1, \dots; 0 \leq x \leq 1\}$$

are monotone and  $M_1 \cup M_2$  is a maximal monotone set which contains both  $M_1$  and  $M_2$ .

In other words,  $M$  is a monotone subset of  $\mathbb{R}^2$  if and only if for any  $[x, u]$  in  $M$ , the set  $M$  lies in the two quadrants

$$Q^+(x,u) = \{[y,v]: x \leq y \text{ and } u \leq v\}, \quad Q^-(x,u) = \{[y,v]: y \leq x \text{ and } v \leq u\}.$$

This clearly implies that by the rotation of angle  $-\pi/4$  around the origin of coordinates the set  $M$  will become the graph of a function (defined on a subset of  $R$ ) satisfying the Lipschitz condition with constant equal to 1. And conversely, the graph of any such function by the rotation of angle  $\pi/4$  will yield a monotone subset of the plane  $R^2$ . It is also clear that if the Lipschitzian function thus associated with a monotone set  $M$  is defined on the whole real line  $R$ , then  $M$  is necessarily maximal monotone. And the Kirszbraun-Valentine theorem implies that also the converse is true: if  $M$  is maximal monotone, then the corresponding function is defined on  $R$ .

In the above examples, the maximal monotone set  $M_1 \cup M_2$  gives by rotation the graph of the function  $f(t) = -\min \{|t-n\sqrt{2}|: n = 0, \pm 1, \dots\}$ . Since in the case of an arbitrary Hilbert space  $X$  the mapping

$$[x,u] \rightarrow \left[ -\frac{1}{\sqrt{2}}(u+x), \frac{1}{\sqrt{2}}(u-x) \right]$$

plays the role of an analogous "rotation", it is clear that Lemma 3.1 and Proposition 3.2 reveal the same relationship between monotone sets and Lipschitzian mappings which is almost trivial and evident for  $X = R$ .

### 3. Monotone mappings

In what follows  $\rightarrow$  will denote the strong convergence in the Banach space  $X$  and  $\rightarrow^*$  the weak\* convergence in

its dual space  $X^*$ . For a set  $C$  in  $X$ ,  $C^0$  will stand for its interior.

Definition 3.2. A mapping  $T:C \rightarrow X^*$  is called hemicontinuous if for any  $x$  in  $C$ ,  $y$  in  $X$  and any sequence  $\{t_n\}$  of positive real numbers, from  $x + t_n y \in C$  ( $n = 1, 2, \dots$ ) and  $t_n \rightarrow 0$  as  $n \rightarrow +\infty$  it follows that  $T(x + t_n y) \rightarrow Tx$ .

If  $C$  is an open or convex set, we can say equivalently that the mapping  $T:C \rightarrow X^*$  is hemicontinuous if it is continuous from line segments in  $X$  to the weak\* topology in  $X$ .

Definition 3.3. A mapping  $T:C \rightarrow X^*$  is called monotone if

$$(3.6) \quad (Tx - Ty, x - y) \geq 0$$

for all  $x, y$  in  $C$ , and strictly monotone if

$$(Tx - Ty, x - y) > 0$$

for all  $x, y$  in  $C$ ,  $x \neq y$ .

Equivalently we can say that the mapping  $T:C \rightarrow X^*$  is monotone if its graph  $\Gamma = \{[x, Tx] : x \in C\}$  is a monotone set in  $X \times X^*$ .

Definition 3.4. A mapping  $T:C \rightarrow X^*$  is called strongly monotone if there exists a continuous positive function  $d(t)$  de-

defined on  $R^+$  with  $\lim d(t) = +\infty$  as  $t \rightarrow +\infty$  such that

$$(3.7) \quad (Tx - Ty, x - y) \geq d(\|x - y\|) \|x - y\|.$$

for all  $x, y$  in  $C$ .

Without loss of generality we may assume that the function  $d(t)$  is strictly increasing, replacing, if necessary,  $d(t)$  by  $d^*(t) = \frac{t}{1+t} \min \{d(s) : t \leq s\}$ . Furthermore, if  $T$  is hemicontinuous, from inequality (3.7) and from

$$\|Tx - Ty\| \|x - y\| \geq (Tx - Ty, x - y)$$

we conclude that necessarily  $d(0) = 0$ .

The following apparently technical lemma states a basic property of hemicontinuous and hemicontinuous monotone mappings.

Lemma 3.2 (Minty [42], Browder [12]). Let  $T: C \rightarrow X$  be a hemicontinuous mapping of a convex subset  $C$  of a Banach space  $X$  into  $X^*$ . Then, for an  $x_0$  in  $C$  and a  $u_0$  in  $X^*$ , the inequality

$$(3.8) \quad (Tx - u_0, x - x_0) \geq 0 \quad \text{for all } x \text{ in } C$$

implies that

$$(3.9) \quad (Tx_0 - u_0, x - x_0) \geq 0 \quad \text{for all } x \text{ in } C.$$

In particular, if (3.8) holds for an  $x_0$  in  $C^0$ , then  $Tx_0 = u_0$ .

If, in addition, the mapping  $T$  is monotone, then (3.8) is equivalent to (3.9).

Proof. If  $T$  is monotone, then

$$(3.10) \quad (Tx - Tx_0, x - x_0) \geq 0 \quad \text{for all } x \text{ in } C$$

so that inequality (3.8) follows from (3.9) by a simple addition of (3.9) to (3.10).

If inequality (3.8) holds, then setting in it  $x_t = (1-t)x_0 + tx$  ( $0 < t \leq 1$ ) in place of  $x$ , we have

$$0 \leq (Tx_t - u_0, t(x - x_0)) = t(Tx_t - u_0, x - x_0).$$

Since  $t > 0$  may be cancelled, we have

$$(Tx_t - u_0, x - x_0) \geq 0.$$

Letting now  $t \rightarrow 0$ , by the weak\* continuity of  $T$  on line segments in  $C$ , we obtain in the limit inequality (3.9).

Finally, if  $x_0$  is an interior point of  $C$ , then inequality (3.9) can obviously hold for all  $x$  in  $C$  if and only if  $Tx_0 - u_0 = 0$ .

Let us observe that if  $T$  is a monotone mapping, inequality (3.8) is equivalent to the assumption that the set  $\Gamma \cup \{[x_0, u_0]\}$ , with  $\Gamma = \{[x, Tx] : x \in C\}$ , is monotone; i.e., that it is a monotone

extension of the monotone set  $\Gamma$ . Therefore, when  $C = X$ , Lemma 3.2 may be interpreted as follows: the graph  $\Gamma$  of a hemicontinuous monotone mapping  $T:X \rightarrow X^*$  is necessarily a maximal monotone subset of  $X \times X^*$ .

It has been already observed that for any monotone subset  $M$  of  $X \times X^*$  there exist maximal monotone sets containing  $M$ . In particular, this is true for the graph  $\Gamma$  of every monotone mapping  $T:C \rightarrow X^*$ . In this case, an important example of such a maximal monotone set is given by the following:

Proposition 3.3 (Browder [12]). Suppose that  $T:C \rightarrow X^*$  is a hemicontinuous monotone mapping of a closed convex subset  $C$  of a Banach space  $X$  into  $X^*$  and assume that the set  $C^\circ$  is nonempty. Then the set

$$G = \{[x, tx+u]: x \in C \text{ and } (u, x-y) \geq 0 \text{ for all } y \text{ in } C\}$$

is a maximal monotone subset of  $X \times X^*$  containing the graph  $\Gamma$  of the mapping  $T$ .

Proof. It is clear that  $\Gamma \subset G$ .  $G$  is a monotone set since if  $[x, Tx+u], [y, Ty+v] \in G$ , then

$$((Tx+u) - (Ty+v), x-y) = (Tx-Ty, x-y) + (u, x-y) + (v, y-x) \geq 0.$$

Suppose now that  $[x_0, u_0] \in X \times X^*$  and

$$(u_0 - u, x_0 - x) \geq 0$$

for all  $[x, u]$  in  $G$ . Without loss of generality we may assume that  $0 \in C^0$ . First of all we assert that  $x_0 \in C$ . Otherwise we would have  $x_0 = sy_0$  for some  $y_0$  on the boundary of  $C$  and some  $s > 1$ . Let  $w_0 \neq 0$  be an element of  $X^*$  such that  $(w_0, y_0 - y) \geq 0$  for all  $y$  in  $C$  and  $(w_0, y_0) > 0$  (such a  $w_0$  certainly does exist since  $y_0$  lies on the boundary of the convex set  $C$  and  $0$  lies in the interior of  $C$ ). By the definition of  $G$ , for every  $t > 0$ ,  $[y_0, Ty_0 + tw_0]$  lies in  $G$ . Hence

$$0 \leq (u_0 - Ty_0 - tw_0, x_0 - y_0) = (s-1)(u_0 - Ty_0 - tw_0, y_0)$$

which is impossible since the right-hand side of this inequality goes to  $-\infty$  as  $t$  goes to  $+\infty$ . This clearly implies that  $x_0 \in C$ .

To complete the proof, observe now that since  $\Gamma \subset G$ , we have

$$(Tx - u_0, x - x_0) \geq 0$$

for every  $x$  in  $C$ . Hence, by Lemma 3.2, for each  $x$  in  $C$ ,

$$(Tx_0 - u_0, x - x_0) \geq 0.$$

Therefore,  $u_0 = Tx_0 + u$  with  $(u, x_0 - x) \geq 0$  for all  $x$  in  $C$ , and this implies that  $[x_0, u_0] \in G$  which completes the proof.



As an immediate consequence of Proposition 3.3 we have the following.

Proposition 3.4 (Browder [14]). Let  $C$  be a closed convex subset of a Banach space  $X$  and  $T:C \rightarrow X^*$  a hemicontinuous mapping. Then for each given  $u_0$  in  $X^*$ , the set

$$S(u_0) = \{x \in C : (Tx - u_0, y - x) \geq 0 \text{ for all } y \text{ in } C\}$$

is convex and closed. In particular, if  $C = X$ , then for each given  $u_0$  in  $X^*$ , the set  $T^{-1}(u_0) = \{x \in X : Tx = u_0\}$  is convex and closed.

Proof. Let  $X'$  be the minimal closed linear subspace of  $X$  containing  $C$ . The interior of  $C$  with respect to  $X'$  is non-empty. It suffices to show, moreover, that the set  $S(u_0)$  is closed in  $X'$ .

For each  $x$  in  $C$ , let  $T'x$  be the restriction of  $Tx$  to the space  $X'$ , i.e. an element of  $(X')^*$ . It is easily seen that the mapping  $T':C \rightarrow (X')^*$  is hemicontinuous and monotone. Furthermore, denoting by  $u'_0$  the restriction of  $u_0$  to  $X'$ , we have

$$S(u_0) = \{x \in C : (T'x - u'_0, y - x) \geq 0 \text{ for all } y \text{ in } C\}.$$

Therefore, without loss of generality we may assume that  $C^0$  is a nonempty subset of  $X$ .

Under this additional assumption, let  $G$  be the maximal

monotone set in  $X \times X^*$  constructed in Proposition 3.3. By the definition of  $G$ ,

$$S(u_0) = \{x \in X : [x, u_0] \in G\}.$$

Hence, by Proposition 3.2,  $S(u_0)$  is a convex closed subset of  $X$ .

Proposition 3.5 (Browder [14]). Let  $C$  be a subset of a Banach space  $X$  and  $T: C \rightarrow X^*$  a monotone hemicontinuous mapping. Then for every weakly compact subset  $D$  of  $C^\circ$ ,  $T(D)$  is a closed subset of  $X^*$ . In particular, if  $X$  is a reflexive Banach space, then for every closed convex and bounded subset  $D$  of  $C^\circ$ ,  $T(D)$  is closed in  $X^*$ .

Proof. Let  $u_0$  be an element of the closure of  $T(D)$ ,

$$u_0 = \lim_{i \rightarrow \infty} Tx_i \quad (x_i \in D \text{ for } i = 1, 2, \dots).$$

By the weak compactness of  $D$ , we may assume that  $x_i \rightarrow x_0$  for some  $x_0$  in  $D$ . Now, for every  $x$  in  $C$ , from the sequence of inequalities

$$(Tx - Tx_i, x - x_i) \geq 0 \quad (i = 1, 2, \dots)$$

in the limit we obtain

$$(Tx - u_0, x - x_0) \geq 0.$$

Hence, by Lemma 3.2, we have  $u_0 = Tx_0$  so that  $u_0$  lies in  $T(D)$ .

The second assertion of Proposition 3.5 follows immediately from the weak compactness of closed convex and bounded subsets of a reflexive Banach space.

The assumption of the monotonicity and hemicontinuity of a mapping  $T$ , when considered for finite dimensional Banach spaces, imply a much stronger continuity property expressed by the following:

Proposition 3.6 (Kato [32]). Let  $T:C \rightarrow X^*$  be a monotone hemicontinuous mapping of a set  $C$  in a finite dimensional Banach space  $X$  into  $X^*$ . Then  $T$  is continuous at every interior point of  $C$ .

Proof. If  $X$  is of finite dimension, then  $X^*$  is also of finite dimension so that in  $X^*$  the weak\* and the strong topologies coincide. Thus we have to show that  $Tx_i \rightarrow Tx_0$  whenever  $x_i \rightarrow x_0$  in  $C^0$ .

First of all we shall show that the sequence  $\{Tx_i\}$  is bounded. Suppose the contrary. Upon passing to a suitable subsequence, if necessary, we may assume without loss of generality that  $s_i = \|Tx_i\|^{-1} \rightarrow 0$  as  $i \rightarrow +\infty$ , and that the sequence  $\{u_i\} = \{s_i Tx_i\}$  is convergent, say, to  $u_0$ . Obviously  $\|u_0\| = 1$  since  $\|u_i\| = 1$  ( $i = 1, 2, \dots$ ). By the monotonicity of  $T$ , for any  $x$  in  $C$ , we have

$$0 \leq s_i (Tx - Tx_i, x - x_i) = (s_i Tx - u_i, x - x_i) \quad (i = 1, 2, \dots).$$

But  $s_i Tx \rightarrow 0$  and  $x_i \rightarrow x_0$  as  $i \rightarrow +\infty$ , so that in the limit  $(u_0, x - x_0) \leq 0$  for all  $x$  in  $C$ . Since  $x_0 \in C^0$ , this implies that  $u_0 = 0$ , in contradiction with  $\|u_0\| = 1$ .

To complete the proof, it suffices now to show that every convergent subsequence of the sequence  $\{Tx_i\}$  is necessarily convergent to  $Tx_0$ . Without loss of generality we may assume that the sequence  $\{Tx_i\}$  itself is convergent, say, to  $u_0$ . Then, by the monotonicity of  $T$ , for every  $x$  in  $C$ , we have

$$(Tx - Tx_i, x - x_i) \geq 0 \quad (i = 1, 2, \dots)$$

and hence, in the limit,

$$(Tx - u_0, x - x_0) \geq 0.$$

Therefore, by Lemma 3.2,  $Tx_0 = u_0$ , and the proof is completed.

In conclusion, let us remark that in general the continuity of a function in a finite dimensional Banach space does not follow from its hemicontinuity as shown by the example of the real-valued function in  $R^2$  defined in the polar coordinates by the formula

$$T(r, \varphi) = r \sin 2\varphi / (\cos^2 \varphi + r^2 \sin^2 \varphi), \quad T(0, 0) = 0$$

which is not continuous at the origin.

4. Examples of monotone mappings

Let  $T$  be a linear mapping of a Banach space  $X$  into its dual space  $X^*$ . For  $T$  inequality (3.6) reduces simply to the following one:

$$(3.11) \quad (Tx, x) \geq 0 \quad \text{for all } x \text{ in } X.$$

In the sequel a linear mapping  $T: X \rightarrow X^*$  will be said to be positive if (3.11) holds, and strictly positive if

$$(Tx, x) > 0 \quad \text{for all } x \text{ in } X, x \neq 0.$$

Similarly, for a linear mapping  $T: X \rightarrow X^*$ , condition (3.7) is simply equivalent to the inequality

$$(Tx, x) \geq d(\|x\|)\|x\| \quad \text{for all } x \text{ in } X.$$

Hence, for  $s$  such that  $sd(s) = 1$ , we have

$$(T(sx/\|x\|), sx/\|x\|) \geq 1 \quad \text{for all } x \text{ in } X, x \neq 0,$$

and this implies that

$$(3.12) \quad (Tx, x) \geq d\|x\|^2 \quad \text{for all } x \text{ in } X$$

with  $d = s^{-2}$ .

A linear mapping  $T: X \rightarrow X^*$  satisfying condition (3.12) will be called strongly positive.

Assume that at an interior point  $x$  of  $C$  the monotone mapping  $T: C \rightarrow X^*$  has the Fréchet derivative, i.e. that there exists a linear continuous mapping  $T_x: X \rightarrow X^*$  such that

$$(3.13) \quad T(x+h) - Tx = T_x h + R_x h \quad (x+h \in C)$$

with  $\|R_x h\| = o(\|h\|)$  as  $\|h\| \rightarrow 0$ . Since for any  $y$  in  $X$ ,  $x + ty$  belongs to  $C$  for  $t$  sufficiently small, from inequality (3.6) and (3.13) we obtain

$$0 \leq (T(x+ty) - Tx, ty) = t^2 [(T_x y, y) + (t^{-1} R_x(ty), y)].$$

Since  $\|t^{-1} R_x(ty)\| = t^{-1} o(t) \rightarrow 0$  as  $t \rightarrow 0$ , we have therefore

$$(T_x y, y) \geq 0 \quad \text{for all } y \text{ in } X.$$

In other words, the Fréchet derivative  $T_x$  of a monotone mapping  $T: C \rightarrow X^*$  at an interior point  $x$  of  $C$  is a positive linear mapping.

Conversely, suppose that the mapping  $T: C \rightarrow X^*$  of a convex subset  $C$  of a Banach space  $X$  into  $X^*$  has a positive Fréchet derivative  $T_x$  at every point  $x$  of  $C$ . Then for any  $x$  and  $x + h$  in  $C$ , the real-valued function

$$\varphi(t) = (T(x+th) - Tx, h)$$

is defined and differentiable in the interval  $[0,1]$ , and nondecreasing since for each  $t$  in  $[0,1]$ ,

$$\varphi'(t) = (T_{x+th}h, h) \geq 0.$$

Therefore, from  $\varphi(0) = 0$  it follows that  $\varphi(1) \geq 0$  which means that

$$(T(x+h)-Tx, h) \geq 0.$$

This clearly implies that  $T$  is a monotone mapping.

The same argument shows that if the Fréchet derivative of a mapping  $T:C \rightarrow X^*$  exists and is strictly positive at every point of a convex set  $C$ , then  $T$  is a strictly monotone mapping.

The above relationship between the monotonicity of a mapping and the positiveness of its Fréchet derivative is a substitute for the classical theorem of analysis which states that a differentiable real-valued function defined in an interval of the real line is monotone if and only if its derivative does not change sign in that interval.

An analogous relationship exists between the convexity of a functional and the positiveness of its Gâteaux derivative (Kačurovskii [29], [30] and Minty [45]).

Let us recall that a functional  $f:X \rightarrow R$  is called convex if

$$f[tx+(1-t)y] \leq tf(x)+(1-t)f(y)$$

for all  $x, y$  in  $X$  and any  $t \in [0, 1]$ . Clearly  $f$  is convex if and only if its restriction to every straight line in  $X$  is a convex function on  $\mathbb{R}$ .

For a given functional  $f: X \rightarrow \mathbb{R}$ , its Gâteaux derivative  $f'(x)$  at a point  $x$  in  $X$  is a linear continuous functional on  $X$  such that

$$(f'(x), h) = \lim_{t \rightarrow 0} \frac{1}{t} (f(x+th) - f(x))$$

for all  $h$  in  $X$ . If the Gâteaux derivative of  $f$  exists everywhere in  $X$ , the mapping  $x \rightarrow f'(x)$  from  $X$  to  $X^*$  is called the gradient mapping of  $f$  and is denoted by  $\text{grad } f$ . Conversely, the mapping  $T: X \rightarrow X^*$  is called potential if it is the gradient of a functional; i.e., if  $T = \text{grad } f$ , for some  $f: X \rightarrow \mathbb{R}$ .

It turns out that any functional  $f: X \rightarrow \mathbb{R}$  with monotone Gâteaux derivative is convex. Indeed, the restriction  $\psi(t) = f(x_0 + th)$  of  $f$  to a straight line  $x = x_0 + th$  ( $-\infty < t < +\infty$ ) is then a differentiable function and

$$\psi'(t) = \lim_{s \rightarrow 0} \frac{1}{s} [f(x_0 + (t+s)h) - f(x_0 + th)] = (f'(x_0 + th), h).$$

Hence, for all  $t_1, t_2$  in  $\mathbb{R}$ ,  $t_1 < t_2$ , we have

$$\begin{aligned} \psi'(t_2) - \psi'(t_1) &= (f'(x_0 + t_2 h), h) - (f'(x_0 + t_1 h), h) \\ &= (t_2 - t_1)^{-1} (f'(x_0 + t_2 h) - f'(x_0 + t_1 h), (t_2 - t_1)h) \geq 0. \end{aligned}$$



Therefore,  $\psi'(t)$  is a nondecreasing function and hence continuous (by the Darboux property of the derivative). This implies that  $\psi(t)$  is a convex function and completes the proof.

Conversely, if  $f: X \rightarrow \mathbb{R}$  is a convex functional with the Gâteaux derivative, then the function  $\psi(t) = f(x_0 + th)$  is differentiable and its derivative  $\psi'(t) = (f'(x_0 + th), h)$  is a nondecreasing function. In particular,  $\psi'(0) \leq \psi'(1)$  which means that

$$(f'(x_0 + h) - f'(x_0), h) \geq 0$$

and this implies that  $f': X \rightarrow X^*$  is a monotone mapping.

Let  $a(x, y)$  be a real-valued function defined in the Cartesian square  $X^2$  of a Banach space  $X$ , linear and continuous with respect to  $y$  for each fixed  $x$ . We can associate with  $a(x, y)$  the mapping  $T: X \rightarrow X^*$  uniquely determined by

$$a(x, y) = (Tx, y) \quad \text{for all } x, y \text{ in } X.$$

It is clear that the mapping  $T$  is monotone if and only if

$$a(x, x-y) - a(y, x-y) \geq 0 \quad \text{for all } x, y \text{ in } X,$$

and strongly monotone if and only if

$$a(x, x-y) - a(y, x-y) \geq d(\|x-y\|)\|x-y\| \quad \text{for all } x, y \text{ in } X$$

and some continuous increasing nonnegative function  $d(t)$ ,  $t \geq 0$ , such that  $\lim_{t \rightarrow +\infty} d(t) = +\infty$ .

### 5. Coercive mappings

Let  $C$  be an arbitrary subset of a Banach space  $X$ .

Definition 3.5. A mapping  $T:C \rightarrow X^*$  is called coercive if

$$(3.14) \quad \lim_{\|x\| \rightarrow +\infty} (Tx, x) / \|x\| = +\infty$$

as  $\|x\| \rightarrow +\infty$ .

Equivalently, a mapping  $T:C \rightarrow X^*$  is coercive if there exists a real-valued continuous function  $c(t)$  defined on  $\mathbb{R}^+$  with  $\lim_{t \rightarrow +\infty} c(t) = +\infty$  and such that

$$(3.15) \quad (Tx, x) \geq c(\|x\|) \|x\| \quad \text{for all } x \text{ in } C.$$

It is clear that the above definition makes sense only if  $C$  is not bounded. It is convenient, however, to call formally coercive every mapping defined only on a bounded subset of  $X$ .

For a linear mapping  $T:X \rightarrow X^*$ , condition (3.15) is equivalent (see the preceding Section) to inequality (3.12), since (3.14) implies that  $c(t) > 0$  for sufficiently large  $t$ . Therefore, for linear mappings the notions of strong positiveness and coerciveness coincide.

It is easily seen that if the mapping  $T:C \rightarrow X^*$  is coercive, then for each  $u$  in  $X^*$  the mapping  $T_u:C \rightarrow X^*$  defined

by  $T_u x = Tx + u$  is also coercive. This follows immediately from the relationship

$$(Tx+u, x) = (Tx, x) + (u, x) = (Tx, x) + o(\|x\|)$$

and condition (3.14).

The condition of coerciveness of a mapping  $T: C \rightarrow X^*$  is basically a condition on the behavior of  $T$  at infinity. Nevertheless, in an implicit manner the zero vector of  $X$  plays in this condition an important role, since condition (3.14) is not invariant under translations in  $X$ . In other words, condition (3.14) does not imply, in general, that

$$(3.16) \quad \lim_{\|x\| \rightarrow \infty} (Tx, x-y)/\|x-y\| = +\infty$$

for every fixed  $y$  in  $X$ .

For instance, let us consider in the plane  $R^2$  the mapping  $T: R^2 \rightarrow R^2$  defined by

$$T: (\xi, \eta) \rightarrow (\xi - \eta \sqrt{\xi^2 + \eta^2}, \xi \sqrt{\xi^2 + \eta^2} + \eta).$$

$T$  is coercive since

$$(T(\xi, \eta), (\xi, \eta)) = \xi^2 + \eta^2.$$

On the other hand, however,

$$(T(\xi, \eta), (\xi, \eta) - (-\eta, \xi) / \sqrt{\xi^2 + \eta^2}) = 0$$

so that for any given point  $(\alpha, \beta)$  on the unit circle, we have

$$(T(\xi, \eta), (\xi, \eta) - (\alpha, \beta)) = 0$$

for all  $(\xi, \eta)$  on the half line with origin at  $(0, 0)$  and forming the angle  $\pi/2$  with the vector  $(\alpha, \beta)$ . Geometrically these properties of the mapping  $T$  are almost obvious, since we obtain  $T$  by defining it first on the half line  $(\xi, 0), \xi > 0$  in such a way that  $T(\xi, 0)$  is perpendicular to the vector  $(\xi, 0) - (0, 1)$  and then extending this definition to the whole plane by a simple repetition of this procedure on every ray issuing from the origin of coordinates.

It turns out, however, that for monotone mappings the condition of coerciveness implies in a certain sense a uniform coerciveness; we have the following:

Proposition 3.7. If  $T: C \rightarrow X^*$  is a coercive monotone mapping, then for any fixed  $y$  in  $C^0$ , relation (3.16) holds true.

Proof. Since  $\|x-y\|/\|x\|$  goes to 1 as  $\|x\|$  goes to infinity, (3.16) is equivalent to

$$\lim_{\|x\| \rightarrow \infty} (Tx, x-y)/\|x\| = +\infty.$$

If  $y \in C$ , then from the inequality

$$0 \leq (Tx - Ty, x - y) = (Tx, x - y) - (Ty, x) + (Ty, y)$$

it easily follows that

$$(3.17) \quad \liminf_{\|x\| \rightarrow \infty} (Tx, x - y) / \|x\| > -\infty.$$

Suppose now that for a  $y_0$  in  $C^0$  and a sequence  $\{x_n\} \subset C$  such that  $\|x_n\| \rightarrow +\infty$  as  $n \rightarrow +\infty$ , we have

$$(3.18) \quad \lim_{n \rightarrow \infty} (Tx_n, x_n - y_0) / \|x_n\| < +\infty.$$

Since  $(Tx_n, x_n) / \|x_n\| \rightarrow +\infty$  as  $n \rightarrow +\infty$ , (3.18) implies that

$$(3.19) \quad \lim_{n \rightarrow \infty} (Tx_n, y_0) = +\infty.$$

For  $s > 0$  sufficiently small,  $y = (1+s)y_0$  belongs to  $C$  so that, from (3.17), (3.18) and (3.19), we have

$$-\infty < \liminf_{n \rightarrow \infty} (Tx_n, x_n - (1+s)y_0) = \lim_{n \rightarrow \infty} (Tx_n, x_n - y_0) -$$

$$- \lim_{n \rightarrow \infty} (Tx_n, y_0) = -\infty$$

and this contradiction completes the proof.

From the proof of Proposition 3.7 it is clear that the assumption  $y \in C^0$  may be replaced by the assumption that  $y$  is an interior point of the set  $C \cap F$ , where  $F$  denotes the straight line through the origin and  $y$ . On the other hand, the mapping  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  considered above shows that Proposition 3.7 is no longer true if  $y$  does not belong to  $C$ : the restriction of  $T$  to the ray  $\{(\xi, 0): \xi > 0\}$  is a monotone mapping but the relation (3.16) does not hold for  $y = (0, 1)$ .

Proposition 3.8. If  $0 \in C$ , then every strongly monotone mapping  $T: C \rightarrow X^*$  is coercive.

Proof. Setting  $y = 0$  in the strong monotonicity condition (3.7) we obtain the coerciveness condition (3.15) with  $c(t) = d(t)$ .

## 6. Inequalities for monotone mappings

The main result of the theory of monotone mappings is that every coercive monotone mapping  $T: X \rightarrow X^*$  of a reflexive Banach space  $X$  into its dual space  $X^*$  is necessarily surjective, i.e. maps  $X$  onto  $X^*$ . In a highly refined and localized form this basic property of monotone mappings is expressed by the following fundamental:

Theorem 3.1 (Browder [12], Hartmand and Stampacchia [26]).

Let  $C$  be a closed convex subset of a reflexive Banach space  $X$  and  $T: C \rightarrow X^*$  a monotone hemicontinuous and coercive mapping. Then for each given  $u_0$  in  $X$ , there exists an  $x_0$  in  $C$  such that

$$(3.20) \quad (Tx_0 - u_0, x - x_0) \geq 0 \quad \text{for all } x \text{ in } C.$$

Before proceeding to the proof of Theorem 3.1, let us make a few preparatory remarks. The mapping  $T': C \rightarrow X^*$  given by  $T'x = Tx - u_0$  for all  $x$  in  $C$  is also hemicontinuous monotone and, as observed in the preceding section, coercive. For this reason, without loss of generality we may consider instead of (3.20) the inequality

$$(3.21) \quad (Tx_0, x - x_0) \geq 0 \quad \text{for all } x \text{ in } C.$$

Furthermore, let  $Y$  be a closed subspace of  $X$  (note that if  $X$  is reflexive, then so is  $Y$  as it easily follows from Theorem 1.4) and let, for every  $x$  in  $C \cap Y$ ,  $T_Y x$  be the restriction of  $Tx$  to  $Y$ . Then  $T_Y$  is a monotone hemicontinuous and coercive mapping of  $C \cap Y$  into  $Y^*$ , since

$$(T_Y x - T_Y y, x - y) = (Tx - Ty, x - y) \geq 0, \quad (T_Y x, x) / \|x\| = (Tx, x) / \|x\|$$

for all  $x, y$  in  $C \cap Y$ . In particular, when  $Y$  is the minimal closed subspace of  $X$  containing  $C$ , then  $T_Y$  is a monotone hemicontinuous mapping of  $C$  into  $Y^*$  and inequality (3.21) is simply equivalent to the following one:

$$(T_Y x_0, x - x_0) \geq 0 \quad \text{for all } x \text{ in } C.$$

This implies that without loss of generality we may assume that  $C$  has interior points in  $X$ .

Finally, if  $y_0$  is an interior point of  $C$ , then the mapping  $\tilde{T}$  defined by  $\tilde{T}x = T(x+y_0)$  maps the set  $\tilde{C} = \{x \in X: x+y_0 \in C\}$  into  $X^*$ , is hemicontinuous monotone and, by Proposition 3.7, coercive. Moreover, inequality (3.21) has a solution in  $C$  if and only if the inequality

$$(\tilde{T}x_0, x-x_0) \geq 0 \quad \text{for all } x \text{ in } \tilde{C}$$

has a solution in  $\tilde{C}$ . Therefore, without loss of generality we may assume that the zero vector of the space  $X$  is an interior point of  $C$ .

Under the additional assumption that  $0 \in C$ , for any solution  $x_0$  of inequality (3.21), we have  $(Tx_0, x_0) \leq 0$ . By the coerciveness of  $T$ , this implies that  $x_0$  lies in the ball  $B(0, \rho)$  with

$$\rho = \inf \{r \in \mathbb{R}^+: (Tx, x) > 0 \text{ for all } x \in C, \|x\| > r\}.$$

Thus, every solution of inequality (3.21) is a solution of the inequality

$$(3.22) \quad (Tx_0, x-x_0) \geq 0 \quad \text{for all } x \text{ in } C \cap B(0, \rho+1).$$

Conversely, if  $x_0$  is a solution of inequality (3.22), then  $\|x_0\| \leq \rho$



so that for any  $x$  in  $C \setminus B(0, \rho+1)$ , the point  $x_0 + (x-x_0)/\|x-x_0\|$  lies in  $C \cap B(0, \rho+1)$  and, therefore,

$$(Tx_0, (x-x_0)/\|x-x_0\|) \geq 0$$

which implies that  $(Tx_0, x-x_0) \geq 0$ . Thus, every solution of inequality (3.22) is necessarily a solution of inequality (3.21). For this reason, without loss of generality we may assume that  $C$  is a bounded subset of  $X$ .

Summing up, to prove Theorem 3.1, we have to prove the existence of a solution to inequality (3.21) under the assumption that  $C$  is a bounded closed and convex subset of  $X$  containing the origin  $0$  in its interior. The proof of this reduced version of Theorem 3.1 will rest upon the following:

Lemma 3.3. Let  $C$  be a bounded closed and convex subset of a finite dimensional Banach space  $X$  with  $0$  in its interior, and let  $T$  be a monotone hemicontinuous mapping of  $C$  into  $X^*$ . Then there exists an  $x_0$  in  $C$  such that  $(Tx_0, x-x_0) \geq 0$  for all  $x$  in  $C$ .

First proof (Browder [12]). We may assume (see Chapter I, Section 6) that  $X$  is a finite dimensional Euclidean space and that  $T$  is a monotone hemicontinuous mapping of  $C$  into  $X$ .

Let  $G$  be the maximal monotone set in  $X^2$  containing the graph of the mapping  $T$ , constructed in Proposition 3.3. Since for

each positive integer  $n$ , the set  $nG = \{[x, nu]: [x, u] \in G\}$  is also maximal monotone, by Proposition 3.2 there exists a sequence

$\{[x_n, Tx_n + u_n]\} \subset C \times X$  such that

$$(3.23) \quad x_n + n(Tx_n + u_n) = 0 \quad (n = 1, 2, \dots).$$

Extracting, if necessary, a suitable subsequence from the sequence  $\{x_n\}$ , we may assume that  $x_n \rightarrow x_0 \in C$  as  $n \rightarrow +\infty$ . From (3.23) it follows immediately that  $Tx_n + u_n \rightarrow 0$  as  $n \rightarrow +\infty$ . Since

$$(Tx - (Tx_n + u_n), x - x_n) \geq 0 \quad \text{for all } x \text{ in } C,$$

in the limit we have

$$(Tx, x - x_0) \geq 0 \quad \text{for all } x \text{ in } C.$$

By Lemma 3.2, therefore, inequality (3.21) holds for all  $x$  in  $C$ , and the proof is completed.

Second proof (Hartman and Stampacchia [26]). As in the first proof, we assume that  $X$  is a Euclidean space.

The interior  $C^\circ$  of the set  $C$  is a union of an infinite sequence  $C_1 \subset C_2 \subset \dots$  of convex closed subsets of  $C^\circ$  such that  $0 \in C_1^\circ$  and for each  $n \geq 1$ , the boundary  $\partial C_n$  has continuous tangent hyperplane. By Proposition 3.6, the mapping  $T$  is continuous in  $C^\circ$  and hence in each  $C_n$  ( $n = 1, 2, \dots$ ).

Suppose now that for every  $n \geq 1$ , there exists in  $C_n$  an  $x_n$  such that

$$(Tx_n, x - x_n) \geq 0 \quad \text{for all } x \text{ in } C_n$$

or, which is the same (see Lemma 3.2), such that

$$(3.24) \quad (Tx, x - x_n) \geq 0 \quad \text{for all } x \text{ in } C_n.$$

Upon passing, if necessary, to a suitable subsequence of  $\{x_n\}$ , we may assume that  $x_n \rightarrow x_0 \in C$  as  $n \rightarrow +\infty$ . Then, from (3.24) in the limit we obtain

$$(Tx, x - x_0) \geq 0 \quad \text{for all } x \text{ in } C^0.$$

By the hemicontinuity of  $T$ , the last inequality also holds for all  $x$  on the boundary of  $C$ , and therefore, again by Lemma 3.2,  $x_0$  is a solution of inequality (3.21).

Thus, to complete the proof, it suffices to prove Lemma 3.3 under the assumption that the mapping  $T: C \rightarrow X$  is continuous (not necessarily monotone) and that the set  $C$  has a continuous tangent hyperplane.

For  $x_0$  in  $\partial C$ , inequality (3.21) is satisfied if and only if

$$Tx_0 = -\lambda Nx_0$$

where  $\lambda \geq 0$  and  $Nx_0$  is the unit normal vector to  $\partial C$  at  $x_0$  directed outward of  $C$ . Therefore, if (3.21) fails to hold for all  $x_0$  in  $\partial C$ , then, for  $0 \leq t \leq 1$ , the continuous vector field

$$T_t x = (1-t)Tx + tNx$$

defined on  $\partial C$ , does not vanish so that the index of  $T$  on  $\partial C$  with respect to  $0 \in C^0$  is equal to the index of the vector field  $N$ . But the latter is different from zero and hence the index  $T$  is also different from zero. This means that the equation  $Tx = 0$  has at least one solution  $x_0$  in  $C^0$  which clearly is a solution of inequality (3.21).

Proof of Theorem 3.1. As observed above, we may restrict ourselves to solve inequality (3.21) under the additional assumption that  $C$  is bounded and contains  $0$  in its interior.

Let  $\mathcal{F}$  be the family of all finite dimensional subspaces of  $X$  ordered (partially) by inclusion. For any  $F$  in  $\mathcal{F}$ , let  $C_F = C \cap F$ . The mapping  $T_F: C_F \rightarrow F^*$ , defined for every  $x$  in  $C_F$  by denoting by  $T_F x$  the restriction of  $Tx$  to  $F$ , satisfies all assumptions of Lemma 3.3. Therefore, for each  $F$  in  $\mathcal{F}$ , the set  $S_F$  of  $x_F$  in  $C_F$  such that

$$(Tx_F, x - x_F) \geq 0 \quad \text{for all } x \text{ in } C_F$$

is nonempty.

For each  $F$  in  $\mathcal{F}$ , denote by  $V_F$  the weak closure of the union of all  $S_{F'}$ , for  $F \subset F'$ ,  $F' \in \mathcal{F}$ . The family  $\mathcal{V} = \{V_F : F \in \mathcal{F}\}$  of weakly closed subsets of the set  $C$  has obviously the finite intersection property. Since  $X$  is reflexive,  $C$  is a weakly compact set. Therefore, there exists in  $C$  an element  $x_0$  which lies in  $V_F$  for all  $F$  in  $\mathcal{F}$ .

Let now  $x$  be an arbitrary element of  $C$  and  $F$  a finite dimensional subspace of  $X$  which contains  $x$ . For every  $x_{F'}$  in  $S_{F'}$  with  $F \subset F'$ , by Lemma 3.2, we have

$$(Tx, x - x_{F'}) \geq 0.$$

Since  $x_0$  lies in  $V_F$ , from this inequality it follows that

$$(Tx, x - x_0) \geq 0,$$

and this inequality holds true for all  $x$  in  $C$ . Again by Lemma 3.2, we have therefore  $(Tx_0, x - x_0) \geq 0$  for all  $x$  in  $C$ , and the proof is completed.

Corollary 3.1. Under the assumptions of Theorem 3.1, for each given  $u_0$  in  $X^*$ , the set  $S(u_0)$  of solutions of inequality (3.20) is a nonempty bounded convex and closed subset of the set  $C$ .

Proof. Theorem 3.1 states that  $S(u_0)$  is nonempty. From its proof it follows that  $S(u_0)$  is bounded. Finally, from Proposition 3.4 it follows that it is closed and convex.

It is also easily seen that if the mapping  $T:C \rightarrow X^*$  is strictly monotone then the solution of inequality (3.20) is unique.

Let us observe that the proof of Theorem 3.1 is nonconstructive in nature. It is not difficult, however, to point out simple but important cases for applications in which it provides us with a method of construction of solution of inequality (3.20). It is so, for instance, when the mapping  $T$  is strictly monotone and continuous and the space  $X$  is separable. In this case we can choose a sequence  $F_1, F_2, \dots$  of finite dimensional subspaces of  $X$  such that  $X$  is the closure of their union. For every  $F_i$  ( $i = 1, 2, \dots$ ) we can then find in the set  $F_i \cap C$  the unique element  $x_i$  such that

$$(Tx - u_0, x - x_i) \geq 0 \quad \text{for all } x \text{ in } F_i \cap C.$$

We claim that the sequence  $\{x_i\}$  is weakly convergent to the unique solution of inequality (3.20). It suffices obviously to show that every weakly convergent subsequence of this sequence is convergent to the solution of (3.20). To this end, suppose that the sequence  $\{x_i\}$  is weakly convergent, say, to  $x_0$ . Then, from the last inequality we obtain

$$(Tx - u_0, x - x_0) \geq 0 \quad \text{for all } x \text{ in } \bigcup_{i=1}^{\infty} F_i \cap C.$$

By the continuity of  $T$ , this inequality holds true for all  $x$  in  $C$ . By Lemma 3.2, this implies that  $x_0$  satisfies inequality (3.20)

and completes the proof.

7. Surjectivity property of monotone mappings

For monotone mappings defined on the whole space  $X$  we easily derive from Theorem 3.1 the following fundamental:

Theorem 3.2 (Minty [43], Browder [5]). Let  $T$  be a monotone hemicontinuous and coercive mapping of a reflexive Banach space  $X$  into its dual space  $X^*$ . Then  $T$  maps  $X$  onto  $X^*$ . For each given  $u_0$  in  $X^*$ , the set  $T^{-1}(x_0) = \{x \in X : Tx = u_0\}$  is bounded closed and convex.

Proof. By Theorem 3.1, for each given  $u_0$  in  $X^*$ , there exists  $x_0$  in  $X$  such that  $(Tx_0 - u_0, x - x_0) \geq 0$  for all  $x$  in  $X$ , and this is possible if and only if  $Tx_0 = u_0$ . The second assertion follows immediately from Corollary 3.1.

For strongly monotone mappings Theorem 3.2 can be considerably strengthened by further information on the inverse mapping.

Theorem 3.3 (Minty [43], Browder [4]). Let  $T$  be a strongly monotone hemicontinuous mapping of a reflexive Banach space  $X$  into its dual space  $X^*$ . Then  $T$  is one-to-one, maps  $X$  onto  $X^*$ , and the inverse mapping  $T^{-1}: X^* \rightarrow X$  is continuous and maps bounded sets of  $X^*$  onto bounded sets of  $X$ .

Proof. By assumption, for a continuous strictly increasing function  $d(t)$  such that  $d(0) = 0$  and  $\lim_{t \rightarrow +\infty} d(t) = +\infty$  as  $t \rightarrow +\infty$ , we have

$$d(\|x-y\|)\|x-y\| \leq (Tx-Ty, x-y) \leq \|Tx-Ty\|\|x-y\| \quad \text{for all } x, y \text{ in } X.$$

Hence

$$d(\|x-y\|) \leq \|Tx-Ty\|$$

so that  $T$  is one-to-one and

$$\|T^{-1}u - T^{-1}v\| \leq d^{-1}(\|u-v\|)$$

for all  $u, v$  in  $R(T) \subset X^*$ . This implies that the mapping  $T^{-1}$  is continuous and maps bounded sets into bounded sets.

Finally, the surjectivity of  $T$  follows directly from Proposition 3.8 and Theorem 3.2.

For strongly positive linear mappings Theorem 3.3 gives the following generalization of the so-called Lax-Milgram lemma (see [35]).

Corollary 3.2. If  $T: X \rightarrow X^*$  is a continuous linear and strongly positive mapping of a reflexive Banach space  $X$  into its dual space  $X^*$ , then  $T$  maps  $X$  onto  $X^*$ .

A direct proof of this corollary runs as follows (Browder [1], [19]). From the inequalities

$$d\|x\|^2 \leq (Tx, x) \leq \|Tx\|\|x\|$$



it follows that  $d\|x\| \leq \|Tx\|$  for all  $x$  in  $X$ . Hence  $T$  is one-to-one and, moreover, its range  $R(T) = \{Tx: x \in X\}$  is a closed linear subspace of  $X^*$ . Indeed, if  $Tx_n \rightarrow u_0$  in  $X^*$ , then the sequence  $\{x_n\}$  is bounded so that upon passing, if necessary, to a suitable subsequence we may assume that the sequence  $\{x_n\}$  is weakly convergent, say, to  $x_0$  in  $X$ . By an easy application of the second version of Theorem 1.2, we conclude that  $Tx_0 = u_0$ .

Suppose now that  $R(T) \neq X^*$ . Then, since  $X^{**} = X$ , there exists in  $X$  a vector  $y \neq 0$  such that  $(Tx, y) = 0$  for all  $x$  in  $X$ . In particular,  $(Ty, y) = 0$ . Hence

$$0 = (Ty, y) \geq d\|y\|^2$$

which yields a contradiction and completes the proof.

In an obvious manner, Theorem 3.2 is equivalent to the statement that if  $T: X \rightarrow X^*$  is a monotone hemicontinuous and coercive mapping of a reflexive Banach space into its dual space, then for each given  $u_0$  in  $X$  the functional equation  $Tx = u_0$  has a solution in  $X$ . By a somewhat more sophisticated argument we can prove the following corollary to Theorem 3.2 which allows us to localize the solution of the equation  $Tx = 0$  and, by an easy modification, the solution of the more general equation  $Tx = u_0$ .

Corollary 3.2 (Minty [43], Browder [5]). Let  $T: X \rightarrow X^*$  be a hemicontinuous monotone mapping of a reflexive Banach space  $X$  into its dual space  $X^*$ , and let

$$(3.25) \quad (Tx, x) \geq 0$$

on the boundary  $\partial C$  of a bounded closed convex subset  $C$  of  $X$  such that  $0 \in C^\circ$ . Then the equation  $Tx = 0$  has a solution in  $C$ .

Proof. From the proof of Theorem 3.1, it is clear that it suffices to prove this statement for finite dimensional Euclidean spaces.

Let  $X$  be a finite dimensional Euclidean space. For any  $\lambda > 0$ , let  $T_\lambda: X \rightarrow X$  be defined by  $T_\lambda x = Tx + \lambda x$  for all  $x$  in  $X$ . Then from inequality (3.25) it follows that

$$(3.26) \quad (T_\lambda x, x) = (Tx, x) + \lambda \|x\|^2 > 0$$

on the boundary  $\partial C$  of  $C$ . By Lemma 3.3, applied to the restriction of  $T_\lambda$  to  $C$ , there exists in  $C$  an element  $x_\lambda$  such that

$$(T_\lambda x_\lambda, x - x_\lambda) \geq 0 \quad \text{for all } x \text{ in } C.$$

Setting in this inequality  $x = 0$ , by confrontation with inequality (3.26) we conclude that  $x_\lambda$  is an interior point of  $C$  which implies, by Lemma 3.2, that  $T_\lambda x_\lambda = 0$ ; i.e.  $Tx_\lambda = -\lambda x_\lambda$ . Now, choosing a sequence  $\{\lambda_n\} \rightarrow 0$  such that the corresponding sequence  $\{x_{\lambda_n}\}$  is convergent, say, to  $x_0$  in  $C$ , we have first

$$Tx_{\lambda_n} = -\lambda_n x_{\lambda_n} \rightarrow 0,$$

and then, by Proposition 3.6,  $Tx_0 = 0$ .

In conclusion, let us observe that the coerciveness condition imposed upon the mapping  $T$  in Theorem 3.2 is essential and cannot be replaced by any weaker condition of similar type. This follows immediately from a simple remark that if  $T$  is a monotone mapping of the real line  $R$  into  $R$ , then the coerciveness condition is equivalent to the assumptions that

$$\lim_{x \rightarrow +\infty} Tx = +\infty \quad \text{and} \quad \lim_{x \rightarrow -\infty} Tx = -\infty,$$

which are indispensable for the surjectivity of the mapping  $T$ .

#### 8. Nonexpansive and monotone mappings in Hilbert spaces

In Hilbert spaces an intimate relationship between monotone and nonexpansive mappings is expressed by the following:

Proposition 3.9 (Minty [42]). Let  $C$  be a subset of a Hilbert space  $X$  and  $U: C \rightarrow X$  a nonexpansive mapping. Then the mapping  $T = I - U$  is monotone.

Proof. It suffices to observe that for all  $x, y$  in  $C$ , we have

$$\begin{aligned} (Tx - Ty, x - y) &= (x - y - (Ux - Uy), x - y) = \|x - y\|^2 - (Ux - Uy, x - y) \\ &\geq \|x - y\|^2 - \|Ux - Uy\| \|x - y\| \geq 0. \end{aligned}$$

It should be noted, moreover, that the mapping  $T$  is

continuous, since  $I$  and  $U$  are continuous mappings.

Proposition 3.9 enables us to give an alternative proof of Proposition 2.5 (Browder [13]). To this end, suppose that the sequence  $\{x_n\} \subset C$  is weakly convergent to  $x_0$  in  $C$ . By Theorem 2.15, we may assume that the nonexpansive mapping  $U$  is defined on the whole space  $X$ . Since  $T = I - U: X \rightarrow X$  is a monotone mapping, for the weakly compact set  $D = \{x_n: n = 1, 2, \dots\}$ ,  $T(D)$  is a closed subset of  $X$  by Proposition 3.5. The assumption that the sequence  $\{x_n - Ux_n\}$  is convergent to  $y_0$  as  $n \rightarrow +\infty$  is then equivalent to the assumption that  $y_0 \in T(D)$ , and this clearly implies that  $Tx_0 = y_0$ .

In a similar way we can derive from Propositions 3.5 and 3.9 an alternative proof of the special case of Theorem 2.2 (Browder [13]): if  $U: C \rightarrow C$  is a nonexpansive mapping of a closed bounded convex subset  $C$  of a Hilbert space  $X$  into  $C$ , then  $U$  has a fixed point in  $C$ . Indeed, without loss of generality we may assume that  $0 \in C$  and that  $U$  is defined in  $X$ . For any  $r \in [0, 1)$ , the mapping  $U_r: C \rightarrow X$  defined by  $U_r x = rUx$  for all  $x$  in  $X$ , is contractive and maps  $C$  into itself. By the Banach contraction principle, for every  $r \in [0, 1)$ , there exists in  $C$  a unique fixed point  $x_r$  of  $U_r$ ;  $x_r = rUx_r$ . Obviously,  $(I - U)x_r = (r - 1)Ux_r = (1 - 1/r)x_r \rightarrow 0$  as  $r \rightarrow 1$ , so that  $0$  lies in the closure of the set  $(I - U)(C)$ . Since, by Proposition 3.5, the set  $(I - U)(C)$  is closed,  $0$  lies in it and this means that  $x_0 - Ux_0 = 0$  for some  $x_0$  in  $C$  and completes the proof.

Let us recall, by the way, that due to Theorem 2.12 we know even more: if  $x_0$  is the fixed point of  $U$  in  $C$  closest to the origin, then  $x_r \rightarrow x_0$  as  $r \rightarrow 1$ .

## 9. Semimonotone mappings

The main results of the theory of monotone mappings can be extended without any considerable changes in the techniques of their proofs to broader classes of mappings whose consideration is motivated by the theory of partial differential equations. In this Section we will be concerned with the extension of that theory to the so-called semicontinuous mappings. For the sake of simplicity we will confine our study to mappings defined on the whole space.

Definition 3.6. A mapping  $T: X \rightarrow X^*$  of a Banach space  $X$  into its dual space  $X^*$  is called semimonotone if there exists a mapping  $S: X^2 \rightarrow X^*$  such that  $Tx = S(x, x)$  for all  $x$  in  $X$  while  $S$  satisfies the three following conditions:

- (i) for each fixed  $y$  in  $X$ , the mapping  $x \rightarrow S(x, y)$  is hemicontinuous;
- (ii) for each fixed  $x$  in  $X$ , the mapping  $y \rightarrow S(x, y)$  is continuous from the weak topology on each weakly compact subset of  $X$  to the strong topology of  $X^*$ ;
- iii) for all  $x, y$  in  $X$ ,

$$(S(x, y) - S(y, y), x - y) \geq 0.$$

Note that every hemicontinuous monotone mapping  $T:X \rightarrow X^*$  is trivially semimonotone with  $S(x,y) = Tx$ , for all  $x,y$  in  $X$ . Similarly, every mapping  $T:X \rightarrow X^*$ , continuous from the weak topology on each compact subset of  $X$  to the strong topology of  $X^*$ , is semimonotone with  $S(x,y) = Ty$  for all  $x,y$  in  $X$ .

The following basic result is a direct generalization of Theorem 3.2.

Theorem 3.4 (Browder [19]). Let  $X$  be a reflexive Banach space and  $T:X \rightarrow X^*$  a semimonotone coercive mapping. Then  $T$  maps  $X$  onto  $X^*$ .

Proof. First of all, by an easy modification of the proof of Proposition 3.6 we can show that every semicontinuous mapping in a finite dimensional space is necessarily continuous. This simply implies that for every  $F$  in the family  $\mathcal{F}$  of all finite dimensional subspaces of  $X$ , the mapping  $T_F:F \rightarrow F^*$  defined for each  $x$  in  $F$  as the restriction of  $Tx$  to  $F$ , is continuous.

By the coerciveness of  $T$ , for a sufficiently large real number  $\rho$ , we have  $(Tx,x) > 0$  on the sphere  $S = S(0,\rho)$ . Therefore, for every  $F$  in  $\mathcal{F}$ ,  $(T_F x, x) > 0$  on the sphere  $F \cap S$ . This enables us, as in the proof of Corollary 3.2, to conclude that for each  $F$  in  $\mathcal{F}$  there exists in the ball  $B = B(0,\rho)$  an element  $x_F$  such that  $T_F x_F = 0$ .

To complete the proof, we shall modify in a simple manner the proof of Theorem 3.1

For each  $F$  in  $\mathcal{F}$ , let  $V_F$  be the weak closure of

the union of all sets  $S_{F'}$  for  $F \subset F'$ ,  $F' \in \mathcal{F}$ ;  $S_F = \{x \in B : T_F x = 0\}$ . The family  $\mathcal{V} = \{V_F : F \in \mathcal{F}\}$  of weakly closed subsets of the ball  $B$  has the finite intersection property. Since  $X$  is a reflexive space,  $B$  is a weakly compact set. Therefore, there exists in  $B$  an element  $x_0$  which lies in  $V_F$  for all  $F$  in  $\mathcal{F}$ .

Let now  $x$  be an arbitrary element of  $C$  and  $F$  a finite dimensional subspace of  $X$  which contains  $x$ . For every  $x_{F'}$  in  $S_{F'}$  with  $F \subset F'$ , we have

$$S(x_{F'}, x_{F'}) = T_{x_{F'}} = T_{F'} x_{F'} = 0.$$

Hence, by condition (iii) of Definition 3.6, we obtain the inequality

$$(S(x, x_{F'}), x - x_{F'}) \geq 0.$$

Since  $x_0$  lies in  $V_F$ , from this inequality and condition (ii) of Definition 3.6 it follows that

$$(S(x, x_0), x - x_0) \geq 0.$$

By condition (i) of Definition 3.6, the mapping  $x \rightarrow S(x, x_0)$  is hemicontinuous. Therefore, Lemma 3.2 implies that  $T_{x_0} = S(x_0, x_0) = 0$  which completes the proof.

#### 10. Densely defined monotone mappings

In this section we will deal with an extension of the surjectivity property of coercive monotone mappings defined in a Banach

space to mappings which are defined only on dense linear subspaces. The simplest result of this kind is stated in the following:

Theorem 3.5 (Browder [16]). Let  $X$  be a reflexive Banach space and  $T$  a hemicontinuous monotone coercive mapping defined on a dense linear subspace  $D$  of  $X$  with values in  $X^*$  such that  $T = L + G$ , where

- (i)  $L$  is a closed linear mapping of  $D(L) = D$  into  $X^*$  such that the adjoint mapping  $L^*$  is the closure of its restriction to  $D(L) \cap D(L^*)$ ;
- (ii)  $G$  is a mapping of  $D$  into  $X^*$  which maps bounded sets of  $X$  into bounded sets of  $X^*$ .

Then  $T$  maps  $D$  onto  $X^*$ .

Proof. It suffices to prove that  $0$  belongs to the range of the mapping  $T$ . To this end, we consider the family  $\mathcal{F}$  of all finite dimensional subspaces of  $D$ , and we define, for every  $F$  in  $\mathcal{F}$ , the mapping  $T_F: F \rightarrow F^*$  as in the proof of Theorem 3.4.  $T_F$  is continuous for each  $F$  in  $\mathcal{F}$  and there exists an element  $x_F$  in  $F$  such that  $T_F x_F = 0$ . Moreover, there is a constant  $\rho$  independent of  $F$  such that  $\|x_F\| \leq \rho$  for each  $x_F$  in

$$S_F = \{x \in F : T_F x = 0\}.$$

By the reflexivity of the space  $X$ , there exists an  $x_0$  in  $X$



such that for every  $F$  in  $\mathcal{F}$ ,  $x_0$  lies in the weak closure of the union of all  $S_{F'}$  with  $F' \supset F$ ,  $F' \in \mathcal{F}$ .

We claim that  $x_0$  belongs to  $D$ . Indeed, let  $x$  be any element of the set  $D(L) \cap D(L^*)$ . For any subspace  $F$  in  $\mathcal{F}$  containing  $x$  and any  $x_F$  in  $S_F$ , we have

$$(3.27) \quad 0 = (T_F x_F, x) = (Tx_F, x) = (Lx_F, x) + (Gx_F, x).$$

On the other hand, by the definition of the adjoint mapping,

$$(3.28) \quad (Lx_F, x) = (L^*x, x_F).$$

Furthermore, from condition(ii) it follows that  $\|Gx_F\| \leq M$  for some constant  $M$  independent of  $F$ . Hence and from (3.27), (3.28), we have

$$|(L^*x, x_F)| \leq M\|x\|.$$

Since the left-hand side of this inequality is weakly continuous in  $x_F$ , we have

$$(3.29) \quad |(L^*x, x_0)| \leq M\|x\| \quad \text{for all } x \text{ in } D(L) \cap D(L^*).$$

From the second part of condition (i) it follows that inequality

(3.29) holds true for all  $x$  in  $D(L^*)$ , and this simply implies

that  $x_0$  belongs to the domain  $D(L^{**})$  of the mapping  $L^{**} = (L^*)^*$ . Since  $L$  is closed,  $D(L^{**}) = D(L)$  so that  $x_0 \in D(L) = D$ .

Now, if  $x$  is an arbitrary element of  $D$ ,  $F$  any finite dimensional subspace of  $D$  containing  $x$  and  $x_F \in S_F$ , then we have

$$0 \leq (Tx - Tx_F, x - x_F) = (Tx - T_F x_F, x - x_F) = (Tx, x - x_F),$$

and hence

$$(Tx, x - x_0) \geq 0 \quad \text{for all } x \text{ in } D.$$

A straightforward application of Lemma 3.2 gives  $Tx_0 = 0$  and completes the proof (it is easily seen that the replacement in that lemma of the whole space  $X$  by its dense linear subspace  $D$  does not affect its validity).

Combining the proof of the last theorem with that of Theorem 3.4 we can obtain the following generalization of both:

Theorem 3.6 (Browder [19]). Let  $X$  be a reflexive Banach space and  $T$  a monotone coercive mapping defined on a dense linear subspace  $D$  of  $X$  with values in  $X^*$  such that  $T = L + G$ , where

- (i)  $L$  is a closed linear mapping of  $D$  into  $X^*$  such that the adjoint mapping  $L^*$  is the closure of its restriction to  $D(L) \cap D(L^*)$ ;

(ii)  $G$  is a mapping of  $X$  into  $X^*$  given by  $Gx = H(x, x)$ , where  $H$  is a mapping of  $X^2$  into  $X^*$  such that for fixed  $y$  in  $X$ ,  $H(\cdot, y)$  is continuous from line segments in  $X$  to the weak topology in  $X^*$ , and for fixed  $x$  in  $X$ ,  $H(x, \cdot)$  is continuous on bounded sets from the weak topology in  $X$  to the strong topology of  $X^*$ ;  $G$  maps bounded sets of  $X$  into bounded sets of  $X^*$ .

Then  $T$  maps  $D$  onto  $X^*$ .

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C h a p t e r   I V  
C O M P L E X   M O N O T O N E   M A P P I N G S

The concept of complex monotone mappings takes its origin from a paper by Zarantonello [6] who has considered continuous mappings  $T$  of a complex Hilbert space  $H$  into itself satisfying the inequality

$$|(Tx - Ty, x - y)| \geq c \|x - y\|^2$$

for some positive constant  $c$  and all  $x, y$  in  $H$ , and has shown that if, in addition,  $T$  maps bounded sets into bounded sets, then  $T$  maps  $H$  onto itself. This result has been strengthened and extended to complex Banach spaces by Browder in [1]-[5].

1. Complex monotone and complex coercive mappings

Let  $X$  be a Banach space over the field of complex numbers and  $X^*$  its dual space. Let  $C$  be an arbitrary subset of  $X$ .

Definition 4.1. A mapping  $T: C \rightarrow X^*$  is called complex monotone if for each positive integer  $N$  there exists a continuous strictly increasing real function  $d_N$  on  $R^+$ ,  $d_N(0) = 0$ , such that

$$(4.1) \quad |(Tx - Ty, x - y)| \geq d_N(\|x - y\|) \|x - y\|$$

for all  $x, y$  in  $C$  with  $\|x\|, \|y\| \leq N$ .

Definition 4.2. A mapping  $T: C \rightarrow X^*$  is called strongly complex monotone if there exists a continuous strictly increasing

function  $d$  on  $\mathbb{R}^+$ ,  $d(0) = 0$ ,  $\lim_{t \rightarrow +\infty} d(t) = +\infty$  as  $t \rightarrow +\infty$ , such that

$$(4.2) \quad |(Tx - Ty, x - y)| \geq d(\|x - y\|)\|x - y\|$$

for all  $x, y$  in  $C$ .

It is clear that any strongly complex monotone mapping is complex monotone.

Definitions 4.1 and 4.2 apply formally to mappings in real Banach spaces. It is easily seen, however, that in the most interesting and important case when  $C$  is a convex set and the mapping  $T: C \rightarrow X^*$  is hemicontinuous, these definitions give nothing new in comparison with the definitions of strictly or strongly monotone mappings. Indeed, suppose, for instance, that condition (4.2) is satisfied by a hemicontinuous mapping  $T: X \rightarrow X^*$ , where  $X$  is a real Banach space of dimension greater than 1. The diagonal  $\Delta = \{[x, x]: x \in X\}$  does not disconnect the space  $X^2 = X \times X$ . Assume that for an element  $[x_0, y_0]$  of  $X^2 \setminus \Delta$ ,  $(Tx_0 - Ty_0, x_0 - y_0) > 0$ . Every other element  $[x_1, y_1]$  of  $X^2 \setminus \Delta$  can be connected in  $X^2 \setminus \Delta$  with  $[x_0, y_0]$  by a polygonal line. By the hemicontinuity of  $T$ , the function  $(Tx - Ty, x - y)$  is continuous on this line. Since it does not vanish,  $(Tx_1 - Ty_1, x_1 - y_1)$  is also positive. This implies that the mapping  $T$  is strongly monotone. If  $(Tx_0 - Ty_0, x_0 - y_0) < 0$ , then the mapping  $-T$  is strongly monotone.

It is easy to see that the same argument shows that if for

a hemicontinuous mapping  $T: X \rightarrow X^*$  defined on a real Banach space  $X$  condition (4.1) holds, then either  $T$  or  $-T$  is strictly monotone, and that the same is true if  $T$  is defined on a convex subset  $C$  of  $X$  with interior points.

Definition 4.3. A mapping  $T: C \rightarrow X^*$  is called complex coercive if there exists a continuous strictly increasing function  $c$  on  $R^+$ ,  $\lim c(t) = +\infty$  as  $t \rightarrow +\infty$ , such that

$$|(Tx, x)| \geq c(\|x\|)\|x\|$$

for all  $x$  in  $C$ .

It is easily seen that if the mapping  $T: C \rightarrow X^*$  is strongly complex monotone and if  $0 \in C$ , then  $T$  is complex coercive. Indeed, setting in inequality (4.2)  $y = 0$ , we obtain

$$|(Tx - T0, x)| \geq d(\|x\|)\|x\|$$

and hence

$$|(Tx, x)| \geq \{d(\|x\|) - \|T0\|\}\|x\|.$$

Proposition 4.1. If the mapping  $T: C \rightarrow X^*$  is complex monotone and complex coercive, then  $T$  is one-to-one and its inverse mapping  $T^{-1}$  is continuous in the range  $R(T)$  of  $T$ .

Proof. If  $T$  satisfies condition (4.1), then  $T$  is one-to-one. Furthermore, for  $x$  in  $C$ , we have

$$\|Tx\|\|x\| \geq |(Tx, x)| \geq c(\|x\|)\|x\|$$

and hence  $\|Tx\| \geq c(\|x\|)$  for  $x \neq 0$ . Since  $c(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ , this implies that the inverse mapping  $T^{-1}$  maps bounded sets in  $R(T)$  into bounded sets in  $C$ . Therefore, for each positive integer  $n$ , there exists a positive integer  $N(n)$  such that if  $u, v \in R(T)$  and  $\|u\|, \|v\| \leq n$ , then  $\|T^{-1}u\|, \|T^{-1}v\| \leq N$ . From inequality (4.1) we have

$$\|T^{-1}u - T^{-1}v\|\|u - v\| \geq |(u - v, T^{-1}u - T^{-1}v)| \geq d_N(\|T^{-1}u - T^{-1}v\|)\|T^{-1}u - T^{-1}v\|.$$

Hence

$$\|T^{-1}u - T^{-1}v\| \leq d_N^{-1}(\|u - v\|).$$

This implies that the mapping  $T^{-1}$  is continuous in the set  $\{u \in R(T) : \|u\| \leq n\}$ .

## 2. Surjectivity property of complex monotone mappings

For complex monotone mappings in reflexive complex Banach spaces we have the following result analogous to Theorem 3.2.

Theorem 4.1 (Browder [5]). Let  $T$  be a complex monotone

and complex coercive mapping of a reflexive Banach space  $X$  into its dual space  $X^*$ , continuous from finite dimensional subspaces of  $X$  to the weak topology of  $X^*$ . Then  $T$  is one-to-one, has continuous inverse and maps  $X$  onto  $X^*$ .

In virtue of Proposition 4.1 it remains only to prove that  $T$  maps  $X$  onto  $X^*$ . The proof of this assertion will rest upon the following:

Lemma 4.1. Theorem 4.1 holds true if  $X$  is of finite dimension.

Proof. The mapping  $T$  is continuous by hypothesis. The mapping  $T^{-1}$  exists and is continuous, by Proposition 4.1. Since  $X^*$  is of the same dimension as  $X$ , by the Brouwer theorem on invariance of domain for mappings in Euclidean spaces, the range  $R(T)$  is an open subset of  $X$ . On the other hand,  $R(T)$  is closed in  $X^*$ , since if  $Tx_n \rightarrow u_0$ , then the sequence  $\{x_n\}$  is bounded, by the coerciveness of  $T$ , and we can suppose that  $x_n \rightarrow x_0$  for some  $x_0$  in  $X$ ; by the continuity of  $T$ , we have  $Tx_0 = u_0$  so that  $u_0 \in R(T)$ . Therefore  $R(T) = X^*$ .

Proof of Theorem 4.1. We have to prove that for every  $u$  in  $X$ ,  $Tx = u$  for some  $x$  in  $X$ . It suffices to prove, however, that this is true for  $u = 0$  since the assumptions of Theorem 4.1 are invariant under passing from the mapping  $T$  to the mapping  $T_u$  defined by  $T_u x = Tx - u$ .

As in the proof of Theorem 3.1, let  $\mathcal{F}$  be the family of

all finite dimensional subspaces of  $X$ , partially ordered by inclusion. For each  $F$  in  $\mathcal{F}$ , let the mapping  $T_F: F \rightarrow F^*$  be defined, for every  $x$  in  $F$ , as the restriction of  $Tx$  to  $F$ .  $T_F$  is continuous, complex monotone and complex coercive, since for all  $x, y$  in  $F$ , we have

$$|(T_F x - T_F y, x - y)| = |(Tx - Ty, x - y)| \geq d_N(\|x - y\|)\|x - y\|$$

for  $N \geq \max(\|x\|, \|y\|)$ , and

$$|(T_F x, x)| = |(Tx, x)| \geq c(\|x\|)\|x\|.$$

By Lemma 4.1,  $T_F$  is one-to-one and maps  $F$  onto  $F^*$ . There exists, therefore, in  $F$  a unique solution  $x_F$  of the equation

$$T_F x = 0.$$

From the coerciveness of  $T$ , we have

$$0 = |(T_F x_F, x_F)| = |(Tx_F, x_F)| \geq c(\|x_F\|)\|x_F\|.$$

Since  $c(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ , there exists an integer  $M$  such that for each  $F$  in  $\mathcal{F}$ ,  $x_F$  lies in the ball  $B = B(0, M)$ .

By hypothesis,  $X$  is reflexive. There exists, there-

fore, an element  $x_0$  in  $X$  such that, for every finite dimensional subspace  $F$  of  $X$ ,  $x_0$  lies in the weak closure  $V_F$  of the set

$$S_F = \{x_F, : F \subset F', F' \in \mathcal{F}\}.$$

In addition, if  $F$  and  $F'$  are in  $\mathcal{F}$  and  $F \subset F'$ , then

$$d_M(\|x_F - x_{F'}\|) \|x_F - x_{F'}\| \leq |(T_F, x_F - T_{F'} x_{F'}, x_F - x_{F'})| = |(Tx_F, x_F - x_{F'})|.$$

In other words, we have

$$d_M(\|x_F - x\|) \|x_F - x\| \leq |(Tx_F, x_F - x)|$$

for every  $x$  in  $S_F$ . The function  $\|x_F - x\|$  is weakly lower semi-continuous in  $x$  and the function  $(Tx_F, x_F - x)$  is weakly continuous. Therefore, since  $x_0$  lies in the weak closure  $V_F$  of  $S_F$ , we have

$$(4.3) \quad d_M(\|x_F - x_0\|) \|x_F - x_0\| \leq |(Tx_F, x_F - x_0)|.$$

In particular, if  $F$  contains  $x_0$ , then

$$(Tx_F, x_F - x_0) = (T_F x_F, x_F - x_0) = 0$$

and hence, by (4.3),  $\|x_F - x_0\| = 0$ ; i.e.,  $x_F = x_0$ .

Now, for every  $x$  in  $X$  and  $F$  containing both  $x_0$  and  $x$ , we have



$$(Tx_0, x) = (Tx_F, x) = (T_F x_F, x) = (0, x) = 0;$$

i.e.,  $(Tx_0, x) = 0$  for every  $x$  in  $X$ . This clearly implies that  $Tx_0 = 0$  and completes the proof.

As an immediate conclusion from Theorem 4.1 and the remark that every strongly complex monotone mapping of the whole space  $X$  into  $X^*$  is necessarily complex coercive we obtain the following:

Corollary 4.1 (Browder [4]). Let  $T$  be a strongly complex monotone mapping of a reflexive Banach space  $X$  into its dual space  $X^*$  continuous from finite dimensional subspaces of  $X$  to the weak topology of  $X^*$ . Then  $T$  is one-to-one, has continuous inverse and maps  $X$  onto  $X^*$ .

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